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SOME PROPERTIES OF THE QUOTIENT SINGULAR VALUE DECOMPOSITION*1)

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Abstract

A new derivation is given for the Quotient Singular value Decomposition (QSVD) of matrix pair (A, B) having the same number of columns. Certain properties of the quotient singular values are proved. The relation between the QSVD and the SVD is analyzed in some detail.

§1. Introduction

In this paper, we will discuss some properties of the Quotient Singular Value Decomposition (QSVD) of a matrix pair (A, B). The QSVD was first proposed by Van loan[18], who used the name the B-singular value decomposition, and further generalized by Paige and Saunders[12], where the name the generalized singular value decomposition was used. We adopt in this paper the name QSVD in accordance with the standardized nomenclature proposed in [5]. Numerical algorithms for computing the QSVD were developed in [10], [15], [19]. Parallel implementations can be found in [1]. There are quite a few papers discussing applications of the QSVD, for example [3], [9], [11], [14], [17]. As pointed out by Speiser, the QSVD together with matrix-vector multiplication, orthogonal triangular decomposition (QR decomposition) and the SVD forms the core linear algebra operations required in most signal processing problems [13]. Despite all those efforts, there are still some questions associated with the QSVD that deserve further investigation. This paper will analyze some theoretical problems concerning the QSVD: in Section 2, we give a new constructive proof of the QSVD, which, when properly adapted, forms a basis of a numerical algorithm for computing the QSVD2); In Section 3, we propose an algorithm for computing the orthonormal basis of the maximal common row space of two matrices having the same number of columns, and we will show how this problem is intimately connected with the QSVD; we also touch on the problem of computing the orthonormal basis of the maximal common row space of

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2) We will not elaborate on the algorithmic aspect of the QSVD in this paper; the reader is referred to [1], [2], [21] for more details.

an arbitrary number od matrices; in Section 4, we generalize the Eckart-Young-Mirsky matrix approximation theorem to handle the case of the quotient singular values; in Section 5, we analyze the relation between the QSVD and the SVD in some detail. A certain form of generalized inverse of matrices generated by the QSVD will also be discussed.

Notation. We also use the following abbreviations in this paper:

$$r_A = \operatorname{rank}(A), \quad r_B = \operatorname{rank}(B), \quad r_{AB} = \operatorname{rank}\begin{pmatrix} A \\ B \end{pmatrix}.$$

Throughout the paper, matrices are denoted by capitals, vectors by lower case letters. The symbol $R^{m\times n}$ represents the set of $m\times n$ real matrices. $\|\cdot\|$ is the spectrum norm and $\|\cdot\|_F$ the Frobenius norm. The identity matrix of order j is denoted by I_j; we will omit the subscript when the dimension is clear from the context. A zero matrix is denoted by O with various dimensions. We also adopt the following convention for block matrices: whenever a dimension indicating integer in a block matrix is zero, the corresponding block row or column should be omitted, and all expressions and equations in which a block matrix of that block row or block column appears, can be discarded.

§2. A New Constructive Proof of the QSVD

In this section, we will give a constructive proof of the QSVD using the SVD and the Gaussian elimination technique. The presentation of the theorem is a dual and slightly generalized version of Theorem 2.3 in [11], where the case of two matrices having the same number of rows is discussed. The techniques used in our proof are quite different from those in [12] and [18]. Extension of the techniques to handle the case of matrix triplets can be found in [20]. As a further generalization in [6] we have provided a systematic and unified treatment for a tree of generalizations of the SVD for any number of matrices with compatible dimensions.

Theorem 2.1. *Let $A \in R^{m\times n}$ and $B \in R^{p\times n}$ have the same number of columns. Then there exist orthogonal matrices U, V and Q such that*

$$U^T AQ = \Sigma_A(L, O), \quad V^T BQ = \Sigma_B(L, O), \tag{2.1}$$

with

$$\Sigma_A = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} I & O & O \\ O & C & O \\ O & O & O \end{pmatrix} \end{matrix}, \quad \Sigma_B = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} O & O & O \\ O & S & O \\ O & O & I \end{pmatrix} \end{matrix} \tag{2.2}$$

where

$$C = \operatorname{diag}(\alpha_{r+1}, \cdots, \alpha_{r+s}), \quad S = \operatorname{diag}(\beta_{r+1}, \cdots, \beta_{r+s}),$$

and