Journal of Computational Mathematics, Vol.11, No.1, 1993, 73–76.

THE CONVERGENCE OF MULTIGRID METHODS FOR NONSYMMETRIC ELLIPTIC VARIATIONAL INEQUALITIES*1)

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Abstract

This paper is concerned with the convergence of multigrid methods (MGM) on nonsymmetric elliptic variational inequalities. On the basis of Wang and Zeng's work (1988), we develop the convergence results of the smoothing operator (i.e. PJOR and PSOR). We also extend the multigrid method of J.Mandel (1984) to nonsymmetric varitional inequalities and obtain the convergence of MGM for these problems.

§1. The Multigrid Algorithm

Let us consider the complementary form of elliptic variational inequalities with the domain $\Omega \in R^s$:

$$\begin{cases} Lu(x) \geq f(x), u(x) \geq c(x), & x \in \Omega, \\ (u(x)-c(x))(Lu(x)-f(x)) \geq 0, & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega. \end{cases} \tag{1.1}$$

Take the sequence of gridlengths $h_0 > h_1 > \cdots > h_l$, the approximation sequence of $\Omega : G^0 \subset G^1 \subset \cdots \subset G^l$, and the corresponding grid number N_k of G^k. We can discretize (1.1) on G^k as follows:

$$\begin{cases} A^k u^k \geq b^k, u^k \geq c^k, \\ (u^k - c^k)^T (A^k u^k - b^k) = 0. \end{cases} \tag{1.2}$$

Wang and Zeng gave a multigrid algorithm in [5]; the main steps are as follows:

(1) ν_1-smooth: $u_1^{\nu,j} = RELAX^{\nu_1}(u_1^j, A^l, b^l)$,

(2) Correcting: $\tilde{u}_l^{\nu,j} = \tilde{u}_l^{\nu,j} + I_{l-1}^l \tilde{u}_{l-1}$, where $\tilde{u}_{l-1}$ is the $(l-1)$-th grid iterative solution of the defect inequality;

(3) ν_2-smooth: $u_l^{j+1} = RELAX^{\nu_2}(\tilde{u}_l^{\nu,j}, A^l, b^l)$.

* Received December 1, 1990.
1) The Project Supported by National Science Foundation of China.

For the sake of convenience, we let $h_1 = h, h_0 = H$ in the case of TWG and rewrite (1.2) and the defect inequality respectively as follows:

$$(\mathrm{P_h})\qquad \begin{cases} A_h u_h \geq b_h, u_h \geq c_h, & (1.3)\\ (u_h - c_h)^T (A_h u_h - b_h) = 0, & (1.4)\end{cases}$$

$$(\mathrm{P_H})\qquad \begin{cases} A_H u_H \geq d_H = (I_H^h)^T (f_h - A_h u_h^{\nu,j}), u_H \geq c_H, & (1.5)\\ (u_H - c_H)^T (A_H u_H - d_H) = 0, & (1.6)\end{cases}$$

where $I_H^h = (p_{ij}) \geq 0$ and, for some positive constant α_h,

$$u_h{}^T A_h u_h \geq \alpha_h u_h^T u_h, \quad \forall u_h \in R^{N_1}, \tag{1.7}$$

$$A_H = (I_H^h)^T A_h I_H^h, \tag{1.8}$$

$$Null(I_H^h) = \{0\}, \tag{1.9}$$

$$c_{HK} = \max\{c_{hi} - u_{hi}^{\nu,j}, p_{ik} \geq 0\}. \tag{1.10}$$

§2. The Convergence of PJOR and PSOR

In this section, we assume that A is a strictly diagonally dominant or irreducible and weakly diagonally dominant matrix, and solve the following problem (P) by relaxation iteration:

$$\begin{cases} Ax \geq b, x \geq c, & (2.1)\\ (x - c)^T (Ax - b) = 0. & (2.2)\end{cases}$$

Let $A = D(I - L - U) = D(I - B)$, where D, L and U are diagonal , strictly lower, and upper triangular matrices respectively. It is not difficult to show that $\rho(|B|) < 1$, where $|B| = (|b_{ij}|)$. Morever we have

Lemma 2.1. *Let x^* be the solution of* (P). *Then*

$$x^* = \max\{c_i, (1-\omega)x_i^* + \omega(D^{-1}b + Lx^* + Ux^*)_i\}. \tag{2.3}$$

Theorem 2.1. *For $0 < \omega < 2/[1 + \rho(|B|)]$, we have*

$$\|\varepsilon^{k+1}\| \leq \|J_\omega^k\| \cdot \|\varepsilon_0\| \tag{2.4}$$

and

$$J_\omega^k \to 0, \quad as\ k \to \infty \tag{2.5}$$

where ε^k is the iterative error of PJOR *and $J_\omega = |1-\omega|I + \omega|B|$, $\|\cdot\| = \|\cdot\|_2$.*

Proof. From Lemma 2.1, it is easy to verify that

$$|\varepsilon^{k+1}| \leq J_\omega|\varepsilon^k| \leq J_\omega^{k+1}|\varepsilon^0|. \tag{2.6}$$

which implies (2.4). Furthermore, $\rho(J_\omega) = |1-\omega| + \omega\rho(|B|) < 1$ for $0 < \omega < 2/[1 + \rho(|B|)]$, and this means (2.6) holds.

Theorem 2.2. *For $0 < \omega < 2/[1 + \rho(|B|)]$, we have*

$$\|\varepsilon^{k+1}\| \leq \|L_\omega^K\| \cdot \|\varepsilon_0\|$$