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A HOMOTOPY ALGORITHM FOR SOLVING THE INVERSE EIGENVALUE PROBLEM FOR COMPLEX SYMMETRIC MATRICES*1)

Xu Shu-fang
(*Department of Mathematics, Peking University, Beijing, China*)

Abstract

A homotopy algorithm for solving the inverse eigenvalue problem for complex symmetric matrices is suggested. Some numerical examples are presented.

§1. Introduction

In this paper we shall consider the following inverse eigenvalue problem.

Problem SCG. Given $n+1$ complex $n \times n$ symmetric matrices $A_0, A_1, \cdots, A_n$, and n complex numbers $\lambda_1, \cdots, \lambda_n$, find n complex numbers $c_1, \cdots, c_n$, such that the matrix $A(c) = A_0 + \sum_{k=1}^{n} c_k A_k$ has eigenvalues $\lambda_1, \cdots, \lambda_n$.

Replacing "complex" by "real" in Problem SCG, we obtain the inverse eigenvalue problem for real symmetric matrices, which is called *Problem SRG* for short.

There is a large literature on numerical methods for solving Problem SRG. But, all those methods require choosing an initial value which is sufficiently close to the solution of Problem SRG so as to guarantee the iterative convergence (see [4] and its references for details). In many cases, it often leads to the failure of the algorithms since it is hard to select a valid initial value. Therefore, how to select a valid initial value becomes a very important problem. However, so far as we know, there is no literature on this problem. In this paper, we propose a homotopy algorithm for solving Problem SCG. Theoretically, this method is independent of the selection of an initial value. Numerical experiments also show its handiness in selecting a valid initial value.

The paper is organized as follows. In §2 we construct a homotopy for solving Problem SCG and prove the existence of homotopy paths by using some results in differential topology and topological degree theory. In §3 we devise a homotopy algorithm for solving Problem SCG by following the homotopy paths . In §4 we give some numerical examples.

Notation and Definitions. Throughout this paper we use the following notation. $C^{m\times l}$ is the set of all $m\times l$ complex matrices. C^m is the set of all m-dimensional complex

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column vectors and $C^1 = C$. R^m is the set of all m-dimensional real column vectors and $R^1 = R$. $SC^{n\times n}$is the set of all $n \times n$ complex symmetric matrices. The norm $\| \quad \|$ stands for both arbitrary vector norm and compatible matrix norm. The superscript T is for transpose. I is the $n \times n$ identity matrix. e_i is the ith column of I. S_n denotes the set of all permutations of $\{1, \cdots, n\}$.

For arbitrary $x = (x_1, \cdots, x_n)^T \in C^n$, we use $D(x)$ to denote the diagonal matrix diag $(x_1, \cdots, x_n)$, i.e., $D(x) = \text{diag}\,(x_1, \cdots, x_n)$. For arbitrary $c = (c_1, \cdots, c_n)^T \in C^n$ and $A_k \in SC^{n\times n}$, $k = 0, 1, \cdots, n$, we define

$$A(c) = A_0 + \sum_{k=1}^{n} c_k A_k.$$

§2. The Construction of Homotopy and Its Properties

Let $A^0, \cdots, A^n \in SC^n$ and $\lambda = (\lambda_1, \cdots, \lambda_n)^T \in C^n$, with $\lambda_i \neq \lambda_j \quad i \neq j$. Define $f : C^{n\times n} \times C^n \longrightarrow C^{n^2+n}$ by

$$f(X,c) = \begin{pmatrix} f_1(X,c) \\ \vdots \\ f_n(X,c) \end{pmatrix} \text{ with } f_i(X,c) = \begin{pmatrix} (A(c) - \lambda_i I)x_i \\ \\ \frac{1}{2}(x_i^T x_i - 1) \end{pmatrix}$$

and $g : C^{n\times n} \times C^n \times C^n \times C^n \longrightarrow C^{n^2+n}$

$$g(X,c,d,\omega) = \begin{pmatrix} g_1(X,c,d,\omega) \\ \vdots \\ g_n(X,c,d,\omega) \end{pmatrix} \text{ with } g_i(X,c,d,\omega) = \begin{pmatrix} (DC - \omega_i I)x_i \\ \\ \frac{1}{2}(x_i^T x_i - 1) \end{pmatrix}$$

where $X = (x_1, \cdots, x_n) \in C^{n\times n}$, $x_i \in C^n$, $i = 1, \cdots, n$, $c = (c_i), d = (d_i), \omega = (\omega_i) \in C^n$, and $D = D(d), C = D(c)$.

A classical result on diagonalizable complex symmetric matrices states that if $B \in SC^{n\times n}$, then B is diagonalizable if and only if there exists a $Q \in C^{n\times n}$ such that

$$B = Q\Lambda Q^T \text{ and } Q^T Q = I$$

where Λ is an $n \times n$ diagonal matrix[5].

So, from the definition of f , it follows that

(1) c^* is a solution of Problem SCG if and only if there exists an $X^* \in C^{n\times n}$ such that $f(X^*, c^*) = 0$.

On the other hand, from the definition of g, we know that

(2) for any $d, \omega \in C^n$ with $d_i \neq 0, i = 1, \cdots, n$, if we define

$$\Gamma_0(d,\omega) = \{(X,c) \in C^{n\times n} \times C^n \quad |g(X,c,d,\omega) = o\},$$

then

$$\Gamma_0(d,\omega) = \{(P_\pi E, \quad c_\pi) | \pi \in S_n, \quad E = \text{diag}\,(\varepsilon_i) \quad \varepsilon_i = 1 \text{ or } -1, \quad 1 \leq i \leq n\} \tag{2.1}$$