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A BLOCK CHAOTIC AND ASYNCHRONOUS ALGORITHM FOR CONSISTENT SYSTEMS WITH INCOMPLETE DATA*

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Abstract

In this paper, we generalize the paracontracting matrices to pseudocontracting matrices. The convergence of (parallel) iteration

$$x_i = P_{j_i, x_{i-1}} x_{i-1}$$

and

$$x_{i+r_i} = \alpha_{j_i} x_{i+r_i-1} + (1-\alpha_{j_i}) P_{j_i, x_i} x_i$$

where $P_{j,x}$, $j = 1, \cdots, n$, are paracontracting and/or pseudocontracting matrices is analyzed. These iterations can also be applied to solve consistent systems with incomplete data.

§1. Introduction

Consider the following system:

$$Rx = f \tag{1.1}$$

where $R \in \mathbb{C}^{n\times m}$, and $m > n$. This sort of systems may arise in application of computed tomography, parallel beam reconstruction, and other fields. In these areas, m may be very large, $m \gg n$. If (1.1) is consistent, its solution set is $\tilde{x} + N(R)$, where $\tilde{x}$ is the unique minimum 2-norm solution of (1.1), and $N(R)$ is the nullspace of R.

Write R into the form:

$$R = \begin{bmatrix} R_1^T \\ \vdots \\ R_n^T \end{bmatrix} \tag{1.2}$$

where $R_i \in \mathbb{C}^m$, and construct paracontracting matrices:

$$P_i = I - \omega \frac{R_i R_i^T}{R_i^T R_i}, \qquad i = 1,\ 2,\ \cdots, n \tag{1.3}$$

where $\omega \in (0,\ 1)$. Based on the Kaczmarz algorithm (c.f. [5]), Elsner et al. proposed an asynchronous paracontracting method to solve (1.1) in [1].

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In this paper, we generalize the algorithm in [1] to the block case. The methods of proving the convergence in [1] are not valid. In Section 2, we present our main results on the block iterations. Then using these results, we propose block and asynchronous algorithms for solving (1.1) in Section 3.

§2. Block Paracontracting and/or Pseudocontracting Iterations

Through this paper, we denote $\|\cdot\|$ as the 2-norm. A^T, x^T denote the Hermitian transpose of the matrix A, vector x, respectively. $R(A)$ denotes the range of A. A matrix $P \in \mathbb{C}^{m\times m}$ is paracontracting (cf. [1], [6]) if

$$Px \neq x \;\Leftrightarrow\; \|Px\| < \|x\| \qquad \text{for all } x \in \mathbb{C}^m\,. \tag{2.1}$$

Lemma 2.1. *If P is paracontracting, for $\forall x \in R(I-P)$, there exists a constant $\gamma < 1$ such that $\|Px\| \le \gamma\|x\|$.*

Let $R \in \mathbb{C}^{n\times m}$, $N = \{1,\cdots,n\}$. N_i, $i = 1, \cdots, p$ are p subsets of N such that $\cup_{i=1}^{p} N_i = N$. Note that we allow $N_i \cap N_j \neq \phi$ for $i \neq j$. n_i is the number of elements in N_i. r_j^T is the jth row of R. R_i is an $m\times n_i$ matrix with columns r_i for all $j \in N_i$. Assume $R_i^T R_i$ is nonsingular, and consider matrices

$$P_i = I - \omega R_i (R_i^T R_i)^{-1} R_i^T \qquad i = 1,\cdots,p. \tag{2.2}$$

In the following, we can prove that P_i is paracontracting. If we need to compute $P_i x$, then we need to solve an equation: $R_i^T R_i c = P_i^T x$. Our first idea is to use some C_i to approximate $R_i^T R_i$:

$$P_i = I - \omega P_i C_i^{-1} R_i^T \qquad i = 1,\cdots,p. \tag{2.3}$$

Theorem 2.2. *Let $P_i = I - \omega R_i C_i^{-1} R_i^T$. $0 < \omega < 1$. If $R_i^T R_i = C_i - (C_i - R_i^T R_i)_i$ is a P-regular splitting, then P_i is paracontracting.*

There is no difficulty to prove Theorem 2.2. We refer the readers to [3] for P-regular splitting. Let $R^T R = D + L + L^T$ where D is diagonal and L is strictly lower triangular. If $C = D$ (Jacobi type), or $C = D + L$ (Gauss-Seidel type), or $C = (D + \lambda L)/\lambda$ with $0 < \lambda < 2$ (SOR type), or $C = D + L + L^T$, then $R^T R = C - (C - R^T R)$ is a P-regular splitting.

The next idea is to choose some number to replace $R_i^T R_i$ in (2.2). Let $P_i = I - \omega\beta_i R_i R_i^T$. In this paper, we use only the contracting property. Then our second idea is: for arbitrary fixed $x \in \mathbb{C}^m$, choose an optimal $\beta_{i,x}$, which depends on x, to minimize $\|P_{i,x}\|$, where $P_{i,x} = I - \beta_{i,x} R_i R_i^T$.

Theorem 2.3. *Let $x \in \mathbb{C}^m$ be arbitrary and fixed. Then*

$$\beta_{i,x} = \begin{cases} (\|R_i^T x\|/\|R_i R_i^T x\|^2, & \text{if } R_i^T x \neq 0 \\ 0, \quad \text{if } R_i^T x \neq 0 \end{cases} \tag{2.4}$$

minimize $\|P_{i,x} x\|$, and $\|P_{i,x} x\| < \|x\| \Leftrightarrow P_{i,x} x \neq x$.