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THE MULTIGRID METHOD FOR TRUNC PLATE ELEMENT*

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Abstract

This paper develops an optimal-order multigrid method for the TRUNC plate element.

This paper will consider the multigrid method for the TRUNC element proposed by Bergan et al. and developed further by Argyris et al. The numerical experiences show that the element has very good convergence(cf. [1,2]). The mathematical proof of convergence of the TRUNC element is also given by Shi Zong-ci in [7]. An optimal multigrid method for the element is given in this paper and the method consists of presmoothing and correction on coarser grids.

§1. The TRUNC Element

Given a triangle K with vertices $a_i = (x_i, y_i), i = 1,2,3$, we denote by λ_i the area coordinates for the triangle and put

$$\xi_1 = x_2 - x_3, \quad \xi_2 = x_3 - x_1, \quad \xi_3 = x_1 - x_2,$$
$$\eta_1 = y_2 - y_3, \quad \eta_2 = y_3 - y_1, \quad \eta_3 = y_1 - y_2.$$

The nodal parameters of the element are the function values and the values of the two first derivatives at the vertices of the triangle K. According to [7], on the triangle K the shape function is an incomplete cubic polynomial,

$$\begin{aligned} w =& b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_1\lambda_2 + b_5\lambda_2\lambda_3 + b_6\lambda_3\lambda_1 + b_7(\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2) \\ &+ b_8(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) + b_9(\lambda_3^2\lambda_1 - \lambda_3\lambda_1^2), \end{aligned} \tag{1.1}$$

which is uniquely determined by the nine nodal parameters $w_i, w_x(i), w_y(i), i = 1,2,3$.

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The coefficients b_i are determined as follows:

$$\begin{cases} b_i = w_i, \quad i = 1,2,3, \\ b_4 = -\frac{1}{2}\{(w_x(1) - w_x(2))\xi_3 + (w_y(1) - w_y(2))\eta_3\}, \\ b_5 = -\frac{1}{2}\{(w_x(2) - w_x(3))\xi_1 + (w_y(2) - w_y(3))\eta_1\}, \\ b_6 = -\frac{1}{2}\{(w_x(3) - w_x(1))\xi_2 + (w_y(3) - w_y(1))\eta_2\}, \\ b_7 = w_1 - w_2 - \frac{1}{2}(w_x(1) + w_x(2))\xi_3 - \frac{1}{2}(w_y(1) + w_y(2))\eta_3, \\ b_8 = w_2 - w_3 - \frac{1}{2}(w_x(2) + w_x(3))\xi_1 - \frac{1}{2}(w_y(2) + w_y(3))\eta_1, \\ b_9 = w_3 - w_1 - \frac{1}{2}(w_x(3) + w_x(1))\xi_2 - \frac{1}{2}(w_y(3) + w_y(1))\eta_2. \end{cases} \tag{1.2}$$

The shape form (1.1) with (1.2) is another Zienkiewicz's element. This element is a C^0 element,, nonconforming for plate bending problems, which converges to the true solution only for very special meshes. The TRUNC element is obtained by modifying the variational formulation.

Let Ω be a convex polygon in $R^2, f \in L^2(\Omega)$. Consider the plate bending problem with the clamped boundary conditions,

$$\begin{cases} \triangle^2 u = f, \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial N}\Big|_{\partial\Omega} = 0. \end{cases} \tag{1.3}$$

The weak form of the problem (1.3) is to find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = (f,v), \quad \forall v \in H_0^2(\Omega), \tag{1.4}$$

where

$$\begin{aligned} a(u,v) &= \int_\Omega (\triangle u \triangle v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}))\,dxdy, \\ (f,v) &= \int_\Omega fv\,dxdy, \end{aligned} \tag{1.5}$$

and $0 < \sigma < \frac{1}{2}$ is the Poisson ratio.

Let $\{\mathcal{K}^k\}_{k=1}^\infty$ be a family of subdivisions of Ω by triangles, where $\mathcal{K}^{k+1}$ is obtained by connecting midpoints of the edges of the triangles in $\mathcal{K}^k$. Let $h_k = \max_{K \in \mathcal{K}^k} \operatorname{diam} K$. Then $h_{k-1} = 2h_k$ and there exist positive constants C_1, C_2, independent of k, such that

$$C_2 h_k^2 \le |K| \le C_1 h_k^2, \quad \forall K \in \mathcal{K}^k, \tag{1.6}$$

where $|K|$ is the area of the triangle K. Throughout the paper, C with or without subscript denotes generic positive constants independent of k.

For $k = 1, 2, \cdots$, defining on each triangle $K \in \mathcal{K}^k$ the shape function in the form of