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IMPLICIT DIFFERENCE METHODS FOR DEGENERATE HYPERBOLIC EQUATIONS OF SECOND ORDER*[1)]

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Abstract

This paper is a sequel to [2]. A two parameter family of explicit and implicit schemes is constructed for the numerical solution of the degenerate hyperbolic equations of second order. We prove the existence and the uniqueness of the solutions of these schemes. Furthermore, we prove that these schemes are stable for the initial values and that the numerical solution is convergent to the unique generalized solution of the partial differential equation.

§1. The Problem and the Difference Schemes

Consider the initial boundary value problem for the degenerate hyperbolic equation of second order

$$\frac{\partial^2 u}{\partial t^2} - x^p a(x,t)\frac{\partial^2 u}{\partial x^2} = b(x,t)\frac{\partial u}{\partial x} + f\Big(x,t,u,\frac{\partial u}{\partial t}\Big), \quad (x,t) \in Q, \tag{1}$$

$$u(x,0) = u_0(x), \quad 0 \le x \le L, \tag{2}$$

$$\partial u/\partial t\,|_{t=0} = u_1(x), \quad 0 \le x \le L, \tag{3}$$

$$u(0,t) \text{ is finite on } 0 \le t \le T, \tag{4}$$

$$u(L,t) = 0, \quad 0 \le t \le T. \tag{5}$$

Here the domain $Q = \{0 < x < L, 0 < t \le T\}$ and $p \ge 1$.

Suppose that the following assumptions are valid for the coefficients in equation (1) and the initial functions:

(A1) $a(x,t)$ is differentiable with respect to x and t on $\bar{Q}$. And there exist constants A_0, A_1, A and C_a, for any $(x,t) \in \bar{Q}$, such that $0 < A_0 \le a(x,t) \le A_1, x^p a(x,t) \le A$, $|\, \partial a/\partial x \,| \le C_a$ and $|\, \partial a/\partial t \,| \le C_a$.

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(A2) $b(x,t)$ is a continuous function of $x \in [0,L]$. Furthermore, there are

$$| b(x,t) | \le B\sqrt{x^p}, \quad (x,t) \in \bar{Q},$$

$$| b(x,t_1) - b(x,t_2) | \le C_b\sqrt{x^p} \, | t_1 - t_2 |, \quad x \in [0,L], \quad t_1, t_2 \in [0,T],$$

where B and C_b are positive constants.

(A3) $f(x,t,u,w)$ is a continuous function of $(x,t,u,w) \in \bar{Q} \times R^2$. And for any $(x,t) \in \bar{Q}$ and $u, w \in R$ there are

$$\left|\frac{\partial}{\partial x} f(x,t,u,w)\right| \le C_f(1+ | u | + | w |), \quad \left|\frac{\partial}{\partial u} f(x,t,u,w)\right| \le C_f$$

where C_f is a positive constant.

(A4) $f(x,t,u,w)$ is semi-bounded for w, i.e., there exists a positive constant C_p such that

$$[f(x,t,u,w_1) - f(x,t,u,w_2)](w_1 - w_2) \le C_p \, | w_1 - w_2 |^2,$$

$$\forall u, w_1, w_2 \in R, (x,t) \in \bar{Q}.$$

(A5) For $x = L$, there is a positive constant C_L such that

$$| f(L,t_1,0,0) - f(L,t_2,0,0) | \le C_L \, | t_1 - t_2 |, \quad t_1, t_2 \in [0,T].$$

(A6) The initial functions satisfy the consistent condition, i.e., $u_0(L) = u_1(L) = 0$. In addition, $u_0(x)$ and $u_1(x)$ are Lipschitz continuously differentiable.

Under the assumptions (A1)–(A6), the existence and uniqueness of the generalized solution of the problem (1)–(5) have been proved by M.L. Krasnov[3].

Solve the problem (1)–(5) by means of the finite difference method. Divide the interval $[0,L]$ and $[0,T]$ into J and $N+1$ parts respectively. The space step is $h = L/J$ and the time step is $k = T/(N+1)$. Let $\omega_k = \{t^n = nk; n = 0,1,\cdots,N+1\}$ and $\omega_h = \{x_j = jh; j = 0,1,\cdots,J\}$. The set of all net points on the domain $\bar{Q}$ is denoted by $\omega_h \times \omega_k$. Let $V(x,t)$ be a discrete function defined on $\omega_h \times \omega_k$ and $V_j^n = V(x_j,t^n)$. Using the same symbols in [2], we denote $V_{\alpha,j}^n, V_{\bar{\alpha},j}^n$ and $V_{\dot{\alpha},j}^n$ as V_j^n's forward, backward and centered difference quotients on variable $\alpha(= x$ or $t)$. The symbols $\|\cdot\|$ and $\|\cdot\|_\infty$, respectively, are L_2 and L_∞ discretized norms with respect to x.

Now, let $V(x,t)$ be the difference approximate solution. We construct a two parameter family of explicit-implicit difference schemes

$$V_{t\bar{t},j}^n - x_j^p a_j^n \bar{V}_{x\bar{x},j}^n = b_j^n \bar{V}_{\dot{x},j}^n + f_j^n, \quad 1 \le j \le J-1, 1 \le n \le N, \tag{6}$$

$$V_{t\bar{t},0}^n = f_0^n, \quad 1 \le n \le N, \tag{7}$$

$$V_J^n = 0, \quad 1 \le n \le N, \tag{8}$$

$$V_j^0 = u_0(x_j), \quad 0 \le j \le J, \tag{9}$$

$$V_{t,j}^1 = u_1(x_j), \quad 0 \le j \le J, \tag{10}$$