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A SHARP ESTIMATE OF A SIMPLIFIED VISCOSITY SPLITTING SCHEME*

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Abstract

A viscosity splitting method for solving the initial boundary value problems of the Navier-Stokes equation, introduced by Zheng and Huang, is considered. We give an improved and sharp estimate in the space $L^\infty(0,T;(L^2(\Omega))^2)$.

§1. Introduction

Let Ω be a bounded domain in R^2. For simplicity we assume that it is a simply connected bounded domain, and its boundary $\partial\Omega$ is sufficiently smooth. Denote by $x=(x_1,x_2)$ a point in R^2. The usual notations $H^s(\Omega), W^{m,p}(\Omega)$ for Sobolev spaces, and $\|\cdot\|_s, \|\cdot\|_{m,p}$ for their norms are applied through out this paper. It is known that $L^2(\Omega)=H^0(\Omega)$.

In [1] the viscosity splitting method for solving the two-dimensional initial boundary value problem of the Navier-Stokes equation

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)u+\frac{1}{\rho}\nabla P=\nu\triangle u+f, \qquad x\in\Omega, t>0, \tag{1.1}$$

$$\nabla\cdot u=0, \qquad x\in\Omega, t>0, \tag{1.2}$$

$$u|_{x\in\partial\Omega}=0, \tag{1.3}$$

$$u|_{t=0}=u_0(x) \tag{1.4}$$

was considered, where $u=(u_1,u_2)$ is the velocity, P is the pressure, the positive constants ν,ρ are the density and viscosity respectively, and ∇ is the gradient, $\triangle=\nabla^2, \nabla\cdot u_0=0, u_o|_{x\in\partial\Omega}=0$. The following scheme was considered: divide the interval $[0,T]$ into equal subintervals with length k; then we solve $\tilde{u}_k(t),\tilde{P}_k(t),u_k(t),P_k(t)$ on each interval $[ik,(i+1)k), i=0,1,\cdots$, according to the following procedure:

First step. Solve a problem on interval $[ik,(i+1)k)$

$$\frac{\partial\tilde{u}_k}{\partial t}+(\tilde{u}_k\cdot\nabla)\tilde{u}_k+\frac{1}{\rho}\nabla\tilde{P}_k=f, \tag{1.5}$$

$$\nabla\cdot\tilde{u}_k=0, \tag{1.6}$$

$$\tilde{u}_k\cdot n|_{x\in\partial\Omega}=0, \tag{1.7}$$

$$\tilde{u}_k(ik)=u_k(ik-0) \tag{1.8}$$

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where n is the unit outward normal vector and $u_k(-0) = u_0$.

Second step. Solve a problem on interval $[ik, (i+1)k)$

$$\frac{\partial u_k}{\partial t} + \frac{1}{\rho} \nabla P_k = \nu \triangle u_k, \tag{1.9}$$

$$\nabla \cdot u_k = 0, \tag{1.10}$$

$$u_k|_{x\in\partial\Omega} = 0, \tag{1.11}$$

$$u_k(ik) = \tilde{u}_k(i+1)k - 0). \tag{1.12}$$

Zheng and Huang proved that this scheme converges, and for any $0 < \varepsilon < \frac{1}{4}$, the rate of convergence is $O(k^{\frac{3}{4}-\epsilon})$ in the space $L^\infty(0,T;(L^2(\Omega))^2)$, where k is the length of the time step.

We now consider the same scheme and give an improved and sharp estimate. Our main result is the following

Theorem. *If* $u_0 \in (H^3(\Omega))^2 \cap (H_0^1(\Omega))^2, \nabla \cdot u_0 = 0,\ f \in L^\infty(0,T;(H^3(\Omega))^2) \cap W^{2,\infty}(0,T;(H^{\frac{1}{2}}(\Omega))^2)$, *$u$ is the solution of problem* (1.1) – (1.4), *$\tilde{u}_k, u_k$ is the solution of problem* (1.5) – (1.12), $0 \le s < 3/2$, *then*

$$\sup_{0\le t\le T} \|\tilde{u}_k(t)\|_{s+1} \le M, \tag{1.13}$$

$$\sup_{0\le t\le T} (\|u(t) - u_k(t)\|_0, \|u(t) - \tilde{u}_k(t)\|_0) \le M'k, \tag{1.14}$$

where the constants M, M' depend only on the domain Ω, constants ν, s, T, and functions f, u_0 and u.

§2. Preliminaries

We will use the Helmholtz operator P and the Stokes operator A frequently. It is known that

$$(L^2(\Omega))^2 = X \oplus G$$

where

$$X = \text{ Closure in } (L^2(\Omega))^2 \text{ of } \{u \in (C_0^\infty(\Omega))^2; \nabla \cdot u = 0\},$$
$$G = \{\nabla P; P \in H^1(\Omega)\}$$

P is the orthogonal projection $P : (L^2(\Omega))^2 \to X$, which is a bounded operator from $H^s(\Omega))^2$ to $(H^s(\Omega))^2$ for any nonnegative s. A is defined as $A = -P\triangle$ with domain $D(A) = X \cap \{u \in (H^2(\Omega))^2; u|_{\partial\Omega} = 0\}$ which admits the following properties:

$$\|A^\alpha e^{-tA}\| \le Ct^{-\alpha}, \qquad \alpha \ge 0, t > 0, \tag{2.1}$$

$$\frac{1}{C}\|u\|_{2\alpha} \le \|A^\alpha u\|_0 \le C\|u\|_{2\alpha}, \qquad \forall u \in D(A^\alpha), \alpha \ge 0 \tag{2.2}$$

and if $0 \le s < \frac{1}{2}$ and $u \in X \cap (H^s(\Omega))^2$, then $u \in D(A^{\frac{s}{2}})$; if $1 \le s < 3/2$ and $u \in D(A) \cap (H^{s+1}(\Omega))^2$, then $u \in D(A^{\frac{s+1}{2}})$.