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A SPECTRAL METHOD FOR A CLASS OF NONLINEAR QUASI-PARABOLIC EQUATIONS*1)

Zhou Zhen-zhong
(Hunan Computing Center, Changsha, China)

Abstract

In this paper, we consider the numerical solution of quasi-parabolic equations of higher order by a spectral method, and propose a computational formula. We give an error estimate of approximate solutions, and prove the convergence of the approximate method and numerical stability on initial values. Under certain conditions, which are much weaker than the conditions in [6], we gain the same convergence rate as in [6].

§1. Introduction

This paper considers the following nonlinear quasi-parabolic equations of higher order with periodic boundary conditions:

$$\frac{\partial}{\partial t}u+(-1)^M A\frac{\partial^{2M+1u}}{\partial x^{2M}\partial t}=f\Big(u,\frac{\partial}{\partial x}u,\frac{\partial^2}{\partial x^2}u,\cdots,\frac{\partial^{2M}u}{\partial x^{2M}}\Big), \tag{1}$$

$$u(x-\pi,t)=u(x+\pi,t),\quad x\in I\!R,\quad 0\le t\le T, \tag{2}$$

$$u(x,0)=\hat{u}_0(x)=\phi(x),\quad x\in I\!R. \tag{3}$$

Here, $u(x,t)$ is a vector function with dimension $J, u(x,t)=(u_1(x,t),\dots,u_J(x,t)), A=(a_{i,j})_{i,j=1}^J$ is a symmetric and positive definite matrix, and a_{ij} are real constants, i.e. $a_{i,j}=a_{j,i}$; $\sum\limits_{i,j=1}^{J}a_{ij}\xi_i\xi_j\ge a_0\sum\limits_{j=1}^{J}\xi_j^2, a_0>0,\forall\xi_j\in I\!R$. Suppose $f\Big(u,\frac{\partial}{\partial x}u,\dots,\frac{\partial^{2M}u}{\partial x^{2M}}\Big)$ in (1) takes the following form:

$$f_j=\sum_{m=1}^{M}(-1)^m D_x^{m+1}\frac{\partial F}{\partial P_{j,m-1}}+\sum_{m=1}^{M}(-1)^m D_x^{m}\frac{\partial G}{\partial P_{j,m-1}}+h_j(u)(j=1,\cdots,J), \tag{4}$$

$$f=(f_1,\cdots,f_J).$$

were $h_j(u)$ is a function of the vector u, the Jacobi matrix of $h=(h_j)_{j=1}^J$ is semi-bounded, i.e. there exists a constant b such that

$$\Big(\xi,\frac{\partial h}{\partial u}\xi\Big)\le b(\xi,\xi),\qquad \forall\xi,\ u\in I\!R,$$

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$F = F(P_0, \dots, P_{M-1}), G = G(P_0, \dots, P_{M-1})$ are smooth functions, $P_M = (P_{1,M}, \dots, P_{J,M}), P_{j,M} = D_x^m u_j, D_x = \frac{\partial}{\partial x}, m = 0, 1, \dots, M-1; j = 1, \dots, J$, and $\hat{u}_0(x)$ is a known vector function with period 2π.

Equations (1) contain many equations which arise in physics and mechanics. For example, when we consider the long wave problems in a nonlinear dispersion system, the BBM equation $u_t + (\phi(u))_x = u_{xxt}$ arises. Also, the nonlinear advective equation of Sobolev-Galpern type is contained in (1).

Zhou and Fu [1] proved that, under some conditions, system (1)—(3) has a generalized solution $u(x,t) \in C^{k+M-1}(I\!R) \cap W_2^{k+M}(-\pi, \pi)(k \geq 1)$. When $k \geq M+1$, there exists a smooth global classical solution, i.e. $D_t D_x^{2M} u(x,t) \in C([0,T] \times I\!R)$.

Since the solution possesses higher smoothness, we can use the spectral method to obtain the numerical solutions of (1)-(3).

We introduce the usual symbols: the inner product of vectors u, v are

$$(u,v) = \int_{-\pi}^{\pi} \sum_{j=1}^{J} u_j v_j dx, \quad \text{set}\Omega = [-\pi, \pi],$$

$$\|u\|_{L_2(\Omega)}^2 = \int_{-\pi}^{\pi} \sum_{j=1}^{J} u_j^2 dx.$$

$L_\infty[0,T; H^S(\Omega)]$ denotes: when $u(x,t)$ as function of x, which belongs to $H^S(\Omega)$ for fixed $0 \leq t \leq T$, and $\sup\limits_{0 \leq t \leq T} \|u(\cdot,t)\|_{H^S} < \infty$; $\|u\|_{L_\infty}[0,T;H^S] = \sup\limits_{0\leq t \leq T} \|u\|_{H^S}, \| \cdot \|H^S$ denotes Sobolev norm, i.e. $\|u\|_{H^S(\Omega)}^2 = \|u\|_{W_2^S(\Omega)}^2 = \sum\limits_{0 \leq |\alpha| \leq S} \left(\frac{\partial^\alpha u}{\partial x^\alpha}, \frac{\partial^\alpha u}{\partial x^\alpha} \right)$; $| \cdot |_{H^S(\Omega)} = \sum\limits_{|\alpha| = S} \left(\frac{\partial^\alpha u}{\partial x^\alpha}, \frac{\partial^\alpha u}{\partial x^\alpha} \right)$, define $\|u\|_{L_\infty(\Omega)} = \text{ess} \sup\limits_{\substack{x\in\Omega \\ 0 \leq j \leq J}} |u_j(x,t)|$. Suppose τ is the time step. We define the difference quotient

$$u_{n+1}, \bar{t} = \frac{1}{\tau}(u(x,(n+1)\tau) - u(x, n\tau)).$$

If there exists a constant b, such that

$$\sum_{j,l=1}^{J} \sum_{m,s=1}^{M} \frac{\partial^2 F}{\partial P_{j,m-1} \partial P_{l,s-1}} \xi_{j,m} \xi_{s,l} \leq b \sum_{j=1}^{J} \sum_{m=1}^{M} \xi_{j,m}^2,$$

then the Hessian matrix of function F is said to be semi-bounded.

2. The Spectral Method of System (1)-(3) and the Priori Integral Estimates of Solutions

We introduce $V_N = \text{span}\{1, \cos x, \sin x, \cdots, \cos Nx, \sin Nx\}$ as a space. P_N denotes