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# PRECONDITIONING OF THE STIFFNESS MATRIX OF LOCAL REFINED TRIANGULATION*1)

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**Abstract**

A preconditioning method for the finite element stiffness matrix is given in this paper. The triangulation is refined in a subregion; the preconditioning process is composed of resolution of two regular subproblems; the condition number of the preconditioned matrix is $O(1+\log\frac{H}{h})$, where $H$ and $h$ are mesh sizes of the unrefined and local refined triangulations respectively.

## 1. Introduction

In practical computation, the triangulation is often refined in a subregion. In this case, the condition number of the stiffness matrix , determined by the mesh size of the local refined triangulation, will be increased seriously.

Let $\Omega\subset R^2$ be a polygonal region, and

$$Lu=-\sum_{i,j=1}^{2}\frac{\partial u}{\partial x_i}(a_{ij}\frac{\partial u}{\partial x_j})+cu$$

be an elliptic operator defined on it, where $(a_{i,j})_{i,j=1,2}$ is symmetric positive definite and bounded from above and below on $\Omega$, $c\ge 0$.

$$\begin{cases} a(u,v)=(f,v), \quad v\in H_0^1(\Omega), \\ u\in H_0^1(\Omega) \end{cases} \tag{1.1}$$

is the variational form of the boundary value problem, with the bilinear form

$$a(u,v)=\int_\Omega\Big[\sum_{i,j=1}^{2}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+cuv\Big].$$

For convenience we discuss only the homogeneous Dirichlet boundary value problem here. The norm in $H_0^1(\Omega)$ introduced by $a(\cdot,\cdot)$ is equivalent to the original one. $H_0^1(\Omega)$ will be treated as a Hilbert space with inner product $a(\cdot,\cdot)$ in the following.

(1.1) is discretized by the finite element method. Triangulation and the linear continuous element will be discussed.

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$\Omega_0 \subset \Omega$ is a polygonal subregion, and $\mathcal{T}$ is a triangulation on $\Omega$, quasi-uniform and locally regular, the mesh size of which is $O(H)$. The boundary of $\Omega_0$ coincides with this triangulation. The triangulation is refined on $\Omega_0$, and we get a triangulation $\mathcal{T}_0$ on $\Omega_0$, where $\mathcal{T}_0$ is quasi-uniform and locally regular on $\Omega_0$, the mesh size of which is $O(h)$. $\mathcal{T}$ and $\mathcal{T}_0$ compose the finite element triangulation.

$S_0^H(\Omega) \subset H_0^1(\Omega)$ is the finite element space corresponding to $\mathcal{T}$ on $\Omega$, $S_0^h(\Omega_0) \subset H_0^1(\Omega_0)$ is the finite element space corresponding to $\mathcal{T}_0$ on $\Omega_0$, and $S = S_0^H(\Omega) + S_0^h(\Omega_0)$ is the finite element space. $\hat{\Omega}$ is the set of finite element node points in $\Omega$ corresponding to $\mathcal{T}$, $\hat{\Omega}_0$ is the set of finite element node points in $\Omega_0$ corresponding to $\mathcal{T}_0$, and $\hat{\Omega} \cup \hat{\Omega}_0$ amounts to the set of finite element node points. $\{\phi_i, i \in \hat{\Omega}\}$ is the usual finite element basis functions of $S_0^H(\Omega)$, and $\{\phi_i^0, i \in \hat{\Omega}_0\}$ is the usual finite element basis functions of $S_0^h(\Omega_0)$. Let

$$\begin{cases} \psi_i = \phi_i, & i \in \hat{\Omega} - \hat{\Omega}_0, \\ \psi_i = \phi_i^0, & i \in \hat{\Omega}_0. \end{cases}$$

The discrete form of (1.1) will be

$$\begin{cases} a(u, \psi_i) = (f, \psi_i), & i \in \hat{\Omega} \cup \hat{\Omega}_0, \\ u \in S. \end{cases} \tag{1.2}$$

The matrix form is

$$Ax = b \tag{1.3}$$

where $A = a(\psi_i, \psi_j)_{i,j \in \hat{\Omega} \cup \hat{\Omega}_0}$. It is well known that Cond $(A) = O(h^{-2})$.

An iterative method is often used to solve (1.3). Preconditioning is an efficient technique to accelerate various iterations. A good preconditioner $Q$ should satisfy the following two conditions: 1) Cond $(Q^{-1}A)$ is small; 2) $Qx = b$ can be solved easily.

Domain decomposition is an important approach of the construction of a preconditioner. In the following, $\Omega_0$ will be decomposed from $\Omega$, and the preconditioning process is composed of resolution of two regular subproblems (discrete problems on quasi-uniform triangulation).

## 2. Construction of the Preconditioner

$$\begin{cases} a(u, \phi_i) = (f, \phi_i), & i \in \hat{\Omega}, \\ u \in S^H \end{cases} \tag{2.1}$$

and

$$\begin{cases} a(u, \phi_i^0) = (f, \phi_i^0), & i \in \hat{\Omega}_0, \\ u \in S^h \end{cases} \tag{2.2}$$

are two discrete problems on $\Omega$ and $\Omega_0$ respectively.

We use $A^H$ and $A^h$ to represent the coefficient matrices of (2.1) and (2.2). (2.1) and (2.2) are regular problems on uniform triangulations. If triangulations $\mathcal{T}$ and $\mathcal{T}_0$ are