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A BETTER DIFFERENCE SCHEME WITH FOUR NEAR-CONSERVED QUANTITIES FOR THE KdV EQUATION*1)

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Abstract

In this paper, we present a new semi-discrete difference scheme for the KdV equation, which possesses the first four near-conserved quantities. The scheme is better than the past one given in [4], because its solution has a more superior estimation. The convergence and the stability of the new scheme are proved.

1. Introduction

The numerical studies of the KdV equation have been largely developed since Zabusky and Kruskal used the second order accuracy Leap-Frog scheme to solve the KdV equation and revealed its important properties [8]. Recently, computational instabilities such as sideband and modulational ones using finite difference approximating were observed by the several scholars[1][6]. Not only that, other numerical methods for the KdV equation also cause the troubles about computational blow-up or numerical spurious solutions when computing time is long. The point is that, even though the KdV equation has infinite conserved quantities, it is very difficult to seek a discretization with more than two discrete conserved quantities.

Consider the periodic boundary problem of the KdV equation

$$u_t + uu_x + u_{xxx} = 0, \quad -\infty < x < +\infty,\ t > 0 \tag{1.1}$$

$$u(x+1,t) = u(x,t), \quad -\infty < x < +\infty,\ t > 0 \tag{1.2}$$

with the initial condition

$$u(x,0) = u_0(x) \tag{1.3}$$

where $u_0(x)$ has period 1 and satisfies adequate smoothness requirements.

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In our previous paper [3], we studied the semi-discrete centered difference scheme

$$V_{jt} + \frac{1}{2}\Delta_0 V_j^2 + \Delta_0 \Delta_+ \Delta_- V_j = 0 \tag{1.4}$$

and proved that the solution of (1.4) has the first three near-conserved quantities and uniform converges to the solution of (1.1) if the initial value $u_0(x)$ is sufficiently smooth. Here, Δ_0, Δ_+ and Δ_- denote, respectively, the centered, the forward and the backward difference quotient operators with respect to space variable x. V_j takes a value on the net point $x_j = jh$ where h is the spatial mesh length such that $Jh = 1$ with a positive integer J. And the meanings of the other symbols in this paper are the same as those in [3] and [9].

In authors' another paper [4], a semi-discrete difference scheme with the first four near-conserved quantities was presented. It can be written as follows:

$$V_{jt} + \frac{1}{2}\Delta_0 V_j^2 + \Delta_0\Delta_+\Delta_- V_j + \frac{1}{6}h^2 V_j \Delta_0\Delta_+\Delta_- V_j - \frac{1}{36}h^4 \Delta_+\Delta_- V_j\, \Delta_0\Delta_+\Delta_- V_j = 0. \tag{1.5}$$

We have proved that the scheme (1.5) is stable and its solution converges to the solution of (1.1) in Sobolev space $\mathbf{L}_\infty(0,T;\mathbf{H}^3)$ for any $T > 0$ if $u_0 \in \mathbf{H}^3$. The four discrete near-conserved quantities are

$$F_0^h(V_h) = \sum_{j=1}^{J} 3V_j h = \text{Const.}, \tag{1.6}$$

$$F_1^h(V_h) = \sum_{j=1}^{J} \frac{1}{2} V_j^2 h = \text{Const.} + O(h^2 t), \tag{1.7}$$

$$F_2^h(V_h) = \sum_{j=1}^{J} (\frac{1}{6}V_j^3 - \frac{1}{2}|\Delta_+ V_j|^2)h = \text{Const.} + O(h^2 t), \tag{1.8}$$

$$\begin{aligned} F_3^h(V_h) &= \sum_{j=1}^{J} \Big\{\frac{5}{72}V_j^4 - \frac{5}{6}V_j(\Delta_+ V_j)^2 - \frac{5}{36}(\Delta_+ V_j)^3 + \frac{1}{2}(\Delta_+\Delta_- V_j)^2\Big\}h \\ &= \text{Const.} + O(h^2 t). \end{aligned} \tag{1.9}$$

In this paper, we give a new scheme which possesses the first four near-conserved (1.6)-(1.8) and (1.10), and a better priori estimate than (1.5).

$$\begin{aligned} F_3^h(V_h) &= \sum_{j=1}^{J} \{\frac{5}{72}V_j^4 - \frac{5}{12}V_j[(\Delta_+ V_j)^2 + (\Delta_- V_j)^2] + \frac{1}{2}(\Delta_+\Delta_- V_j)^2\}h \\ &= \text{Const.} + O(h^2 t). \end{aligned} \tag{1.10}$$

Applying the theory of discrete functional analysis due to Zhou[9] and the technique of coupled priori estimating by the authors[2], we prove the convergence and the stability of the new scheme.