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SYMPLECTIC MULTISTEP METHODS FOR LINEAR HAMILTONIAN SYSTEMS *1)

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Abstract

Three classes of symplectic multistep methods for linear Hamiltonian systems are constructed and their stabilities are discussed in this paper.

1. Introduction

Professor Feng Kang advanced the principle for construction of symplectic algorithms for Hamiltonian systems[1] and pointed out ,that symplectic algorithms can reflect main features of Hamiltonian systems, therefore they are more available. Plenty of theoretical and numerical results have proved these points.

Professor Feng Kang also discussed the approximation problems by algebraic functions. The conclusions are stated as follows[2]:

1. We note $\psi(\xi) = \rho(\xi)/\sigma(\xi)$. A multistep method $M(\rho, \sigma)$ is symplectic for linear Hamiltonian systems (we call it linear symplectic for short) iff $\psi(\xi) = -\psi(1/\xi)$.

2. Assume ρ and σ have no common factor. $\rho(\xi)$ is antisymmetric $(\xi^K \rho(1/\xi) = -\rho(\xi))$ and $\sigma(\xi)$ is symmetric $(\xi^K \sigma(1/\xi) = \sigma(\xi))$ iff $\psi(\xi) = -\psi(1/\xi)$.

Dased on the above results, in this paper three classes of linear symplectic multistep formulas are given and some good proterties are discussed.

2. The Construction of Linear Symplectic Multistep Formulae

Lemma 1. *All roots of $\rho(\xi)$ have module equal to unit and are simple if the linear multistep formulae $M(\rho, \sigma)$ are linear symplectic.*

Proof. According to symplectic condition $\psi(\xi) = -\psi(1/\xi)$ if ξ is a root of $\rho(\xi)$, so does $1/\xi$. The module of no root of $\rho(\xi)$ is exceeds 1 and the roots of module 1 are simple (stability condition), therefore lemma 1 holds.

Lemma 2. *$\rho(\xi)$ is antisymmetric if $\rho(\xi) = (\xi-1)(\xi+1)(\xi-e^{i\varphi_1})(\xi-e^{-i\varphi_1})\cdots(\xi-e^{i\varphi_P})(\xi-e^{-i\varphi_P})$.*

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Proof. Since $\xi(1/\xi - 1) = -(\xi - 1)$, $\xi(1/\xi + 1) = (\xi + 1)$ and

$$\xi^2(1/\xi - e^{i\varphi})(1/\xi - e^{-i\varphi}) = (1 - \xi e^{i\varphi})(1 - \xi e^{-i\varphi}) = (\xi - e^{i\varphi})(\xi - e^{-i\varphi}),$$

we have

$$\xi^{2(p+1)}\rho(1/\xi) = -\rho(\xi).$$

When

$$\rho(\xi) = (\xi - 1)(\xi - e^{i\varphi_1})(\xi - e^{-i\varphi_1})\cdots(\xi - e^{i\varphi_p})(\xi - e^{-i\varphi_p}),$$

the proof is the same.

Theorem 1. *$\rho(\xi)$ stands for above mentioned antisymmetric polynomial of degree k. If k is even, then the only symmetric polynomial $\sigma(\xi)$ of degree k can be defined, so that the linear symplectic implicit k-step formulae $M(\rho, \sigma)$ have order $k + 2$. (i.e. optimal methods). If k is odd, then the only symmetric polynomial $\sigma(\xi)$ of degree k can be defined, so that the linear symplectic implicit k-step formulae $M(\rho, \sigma)$ have order $k + 1$.*

Proof. Let

$$\xi = \frac{1+z}{1-z}, \qquad z = \frac{\xi - 1}{\xi + 1}$$

and

$$r(z) = \Big(\frac{1-z}{2}\Big)^K \rho\Big(\frac{1+z}{1-z}\Big), \qquad s(z) = \Big(\frac{1-z}{2}\Big)^K \sigma\Big(\frac{1+z}{1-z}\Big).$$

$r(z)$ is odd function because

$$r(-z) = \Big(\frac{1+z}{2}\Big)^K \rho\Big(\frac{1-z}{1+z}\Big) = -\Big(\frac{1+z}{2}\Big)^K \Big(\frac{1-z}{1+z}\Big)^K \rho\Big(\frac{1+z}{1-z}\Big)$$
$$= -\Big(\frac{1-z}{2}\Big)^K \rho\Big(\frac{1+z}{1-z}\Big) = -r(z).$$

It is well known that the multistep method $M(\rho, \sigma)$ has order p if and only if $P(z) = r(z)/\log(\frac{1+z}{1-z}) - s(z)$ has a zero of order p at z=0 (see Henrici [3]). We choose the k terms in front of Taylor series of $r(z)/\log(\frac{1+z}{1-z})$ as s(z), then the multistep method $M(\rho, \sigma)$ associated with r(z) and s(z) has order k+1. Both $r(z)$ and $\log((1+z)/(1-z))$ are odd, so s(z) is even. According to this, $\sigma(\xi)$ is symmetric and $M(\rho, \sigma)$ has order k+2 when k is even.

Theorem 2. *$\rho(\xi)$ stands for above mentioned antisymmetric polynomial of degree k. If k is even, then a symmetric polynomial $\sigma(\xi)$ of degree k can be defined, so that the linear symplectic explicit k-step formulae $M(\rho, \sigma)$ have order k.*

Proof. For optimal methods $\sigma(\xi)$ may express as

$$\sigma(\xi) = C_0\xi^{k/2} + C_1\xi^{k/2-1}(\xi - 1)^2 + \cdots + C_{k/2}(\xi - 1)^k \tag{1}$$

where C_i is a definite constant. If the last term $C_{k/2}(\xi - 1)^k$ in (1) is taken away, $\sigma(\xi)$ is a symmetric polynomial of degree $k - 1$ and the corresponding linear symplectic k-step formulae are explicit and of order k. (see [3]).