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MULTIPLICATIVE EXTRAPOLATION METHOD FOR CONSTRUCTING HIGHER ORDER SCHEMES FOR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we develop a new technique called multiplicative extrapolation method which is used to construct higher order schemes for ordinary differential equations. We call it a new method because we only see additive extrapolation method before. This new method has a great advantage over additive extrapolation method because it keeps group property. If this method is used to construct higher order schemes from lower symplectic schemes, the higher order ones are also symplectic. First we introduce the concept of adjoint methods and some of their properties. We show that there is a self-adjoint scheme corresponding to every method. With this self-adjoint scheme of lower order, we can construct higher order schemes by multiplicative extrapolation method, which can be used to construct even much higher order schemes. Obviously this constructing process can be continued to get methods of arbitrary even order.

Introduction

When we construct a higher order scheme for systems of ordinary differential equations:

$$y' = f(y) \tag{1}$$

(where $y = y(x)$, and x is a variable), we often use the "Taylor series expanding" method, but sometimes this method is very tedious when it is applied to get higher order schemes. There is another method: Lie series, it is the method we use in this paper. J.Dragt, F.Neri, and Stanly Steinberg have done a lot of work in developing this method. For details, one can refer to [4,6,8]. We just apply this method to our problem, and do not need to compute out the exact terms of the "Lie series" of a scheme: we just use the form of them. Thus the deduction becomes simple when this Lie series method is applied to multiplicative extrapolating method as we will show later.

In section 1, we will give the definition of adjoint methods, self-adjoint methods and some properties of them. Section 2 is about the multiplicative extrapolation method.

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This paper is a summary of a pre-print one, all the proofs and other details omitted here are given in that paper.

Notice: all the functions considered in this paper are supposed to be analytic.

1. Adjoint Method And Self-adjoint Method

We know every one-step difference scheme can be written as follows:

$$y_{n+1} = s(\tau)y_n \tag{2}$$

where $s(\tau)$ is the operator corresponding to the difference scheme, and τ is the step length.

Definition 1.1. *An operator $s^*(\tau)$ is called the adjoint operator of $s(\tau)$, if*

$$s^*(-\tau)s(\tau) = I \tag{3.1}$$

$$s(\tau)s^*(-\tau) = I \tag{3.2}$$

are satisfied.

We rewrite (2) in the form:

$$y_{n+1} = y_n + \tau\Phi(x, y_n, \tau) \tag{4}$$

here

$$\begin{cases} y_{n+1} = y_\tau(x_n + \tau) \\ y_n = y_\tau(x_n) \end{cases} \tag{5}$$

and $\Phi(x, y_n, \tau)$ is the increment function corresponding to the scheme (2).

Definition 1.2. *A scheme $y_{n+1} = y_n + \tau\Phi^*(x, y_n, \tau)$ is the adjoint of (4) if*

$$B = A - \tau\Phi(x + \tau, A, -\tau) \tag{6.1}$$

$$A = B + \tau\Phi^*(x, B, \tau) \tag{6.2}$$

are satisfied.

Theorem 1.3. *The definitions 1.1 and 1.2 are equivalent.*

Definition 1.4. *We call an operator $s(\tau)$ is self-adjoint, if $s^*(\tau) = s(\tau)$.*

Theorem 1.5. *For any operator $s(\tau)$, $s^*(\tau)s(\tau)$ (or $s(\tau)s^*(\tau)$) is a self-adjoint operator.*

Theorem 1.6. *The symmetric composition $s_1(\tau)s_2(\tau)s_1(\tau)$ of self-adjoint operators $s_1(\tau)$, $s_2(\tau)$ is a self-adjoint operator.*

2. Multiplicative Extrapolation Method for Constructing of Higher Order Integrators

Denote $f = [f_1, f_2, \cdots, f_n]^T, g = [g_1, g_2, \cdots, g_n]^T, D = [\frac{d}{dy_1}, \frac{d}{dy_2}, \cdots, \frac{d}{dy_n}]^T$ where $f_1, f_2, \cdots, f_n$ and $g_1, g_2, \cdots, g_n$ are scalar functions. Let

$$L_f = f^T D = \sum_{i=1}^{n} f_i \frac{\partial}{\partial y_i} \tag{7}$$