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ON STABILITY OF SYMPLECTIC ALGORITHMS*[1)]

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Abstract

The stability of symplectic algorithms is discussed in this paper. There are following conclusions.

1. Symplectic Runge-Kutta methods and symplectic one-step methods with high order derivative are unconditionally critically stable for Hamiltonian systems. Only some of them are A-stable for non-Hamiltonian systems. The criterion of judging A-stability is given.

2. The hopscotch schemes are conditionally critically stable for Hamiltonian systems. Their stability regions are only a segment on the imaginary axis for non-Hamiltonian systems.

3. All linear symplectic multistep methods are conditionally critically stable except the trapezoidal formula which is unconditionally critically stable for Hamiltonian systems. Only the trapezoidal formula is A-stable, and others only have segments on the imaginary axis as their stability regions for non-Hamiltonian systems.

1. Fundamental Definitions

Lemma 1. *The solution of a linear ordinary differential equation with constant coefficient $\dot{Y} = AY$ is stable if all eigenvalues of A have nonpositive real parts and the eigenvalues with null real part are single roots of the minimal polynomial.*

The linear Hamiltonian system can be denoted as $\dot{Z} = JSZ$ where $Z = (\,pq)$, $J = (\,0 - EE0)$, and the Hamiltonian function $H(z) = z'sz2$.

Lemma 2. *The solutions of linear Hamiltonian systems are critically stable if all eigenvalues of JS have null real part and are single roots of the minimal polynomial.*

Definition 1. *When the model equation $\dot{Y} = AY$ is solved using a numerical method, the method is A-stable if its stability region involves the whole left half plane.*

Definition 2. *When the linear Hamiltonian system $\dot{Z} = JSZ$ is solved using a one-step method, the one-step method is critically stable if all eigenvalues of the amplification matrix have module 1 and are single roots of the minimal polynomial.*

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Definition 3. *When the linear Hamiltonian system $\dot{Z} = JSZ$ is solved using a multistep method, the multistep method is critically stable if all roots of the characteristic equation are on the unit circle and are single roots.*

2. The Stability of One-Step Symplectic Algorithms

When the one-step symplectic algorithms (such as symplectic Runge-Kutta methods, the one-step symplectic methods with higher order derivative) are used to solved the model equation $\dot{Y} = AY$, in most cases the characteristic equation always has the following form[1−2]:

$$\xi(\mu) = P(\mu)P(-\mu)$$

where $\mu = h\lambda$, λ stands for the eigenvalues of the matrix A and P is a polynomial with real coefficient.

Theorem 1. *The one-step method is unconditionally critically stable if the corresponding characteristic equation can be expressed as $\xi(\mu) = P(\mu)P(-\mu)$.*

Proof. When $\lambda = iy$, we have

$$\xi = P(\mu)P(-\mu) = P(ihy)P(-ihy) = P(ihy)P(\overline{ihy}) = P(ihy)\bar{P}(ihy)$$

and

$$\overline{\xi} = \bar{P}(ihy)\bar{P}(-ihy) = \bar{P}(ihy)P(ihy).$$

Therefore $\xi \cdot \overline{\xi} = 1$. The above relationship holds as long as λh is on the imaginary axis, so there is no restriction on h.

Theorem 2. *If all poles of $\xi(w)$ are on the right half plane, then the corresponding one-step symplectic method is A-stable.*

Proof. .Unconditional critical stability means $\xi = P(\mu)P(-\mu) = 1$ as long as $\mu = \lambda h = iyh$, where $-\infty < y < \infty$.

If all poles of ξ are on the right half plane, then ξ is an analytical function on the left half plane. According to the maximum module principle we have $\xi(\mu) < 1$ as long as $\mathrm{Re}(\mu) < 0$.

Example 1. The midpoint formula $y_{n+1} = y_n + hf(y_n + y_{n+1}2)$ and trapezoidal formula $y_{n+1} = y_n + h2(f(y_n) + f(y_{n+1}))$ have the same characteristic equation $\xi = 1 + \mu 1 - \mu$, so they are unconditionally critically stable and A-stable. The symplectic schemes, whose characteristic equations are diagonal Pade approximation, are the same (such as s-stage Runge-Kutta methods with order $2s + 2$).

Example 2. The composite symplectic schemes of order 4.[3]

$$\Big\{ y_{n+\frac{1}{3}} = y_n + c_1 h2(f(y_n) + f(y_{n+\frac{1}{3}})), y_{n+\frac{2}{3}} = y_{n+\frac{1}{3}} + c_2 h2(f(y_{n+\frac{1}{3}}) + f(y_{n+\frac{2}{3}})), y_{n+1} = y_{n+\frac{2}{3}} + c_1 h2(f(y_{n+\frac{2}{3}}) + f$$

and

$$\Big\{ y_{n+\frac{1}{3}} = y_n + c_1 hf\Big(y_n + y_{n+\frac{1}{3}}2\Big), y_{n+\frac{2}{3}} = y_{n+\frac{1}{3}} + c_2 hf\Big(y_{n+\frac{1}{3}} + y_{n+\frac{2}{3}}2\Big), y_{n+1} = y_{n+\frac{2}{3}} + c_1 hf\Big(y_{n+\frac{2}{3}} + y_{n+1}2\Big)$$