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FOURIER-LEGENDRE SPECTRAL METHOD FOR THE UNSTEADY NAVIER-STOKES EQUATIONS*

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Abstract

Fourier-Legendre spectral approximation for the unsteady Navier-Stokes equations is analyzed. The generalized stability and convergence are proved respectively.

1. Introduction

The numerical methods of the Navier-Stokes equations can be found in [1-4]. Specific algorithms in [5-8] have been devoted to the semi-periodic cases which describe channel flow, parallel boundary layers, curved channel flow and cylindrical Couette flow. In this paper, we consider the mixed Fourier-Legendre spectral approximation for the unsteady Navier-Stokes equations. We use Fourier spectral approximation in the periodic directions and Legendre spectral approximation in the non-periodic one. For approximating continuity equation, we adopt small parameter technique[9]. This method has better stability and higher accuracy.

Let $x = (x_1, \cdots, x_n)^T \quad (n = 2 \text{ or } 3)$ and $\Omega = I \times Q$ where $I = \{x_1 / -1 < x_1 < 1\}$, $\quad Q = \{y = (x_2, \cdots, x_n)^T / -\pi < x_q < \pi, 2 \le q \le n\}$. We denote by $U(x,t)$ and $P(x,t)$ the speed and the pressure. $\nu > 0$ is the kinetic viscosity. $U_0(x), P_0(x)$ and $f(x,t)$ are given functions. We consider the Navier-Stokes equations as follows

$$\begin{cases} \frac{\partial U}{\partial t} + (U \cdot \nabla)U - \nu \nabla^2 U + \nabla P = f, & \text{in} \quad \Omega \times (0,T], \\ \nabla \cdot U = 0, \qquad \text{in} \quad \Omega \times (0,T], \\ U(x,0) = U_0(x), \quad P(x,0) = P_0(x), \quad \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

Assume that all functions have the period 2π for the variable y. In addition, we also suppose that U satisfies the homogeneous boundary conditions in the x_1-direction

$$U(-1, y, t) = U(1, y, t) = 0, \quad \forall y \in Q.$$

To fix $P(x,t)$, we require

$$\mu(P) \equiv \int_\Omega P(x,t)dx = 0, \quad \forall t \in [0,T].$$

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We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the usual inner product and norm of $L^2(\Omega)$. $|\cdot|_1$ denotes the semi-norm of $H^1(\Omega)$. Let $C^\infty_{0,p}(\Omega)$ be the subset of $C^\infty(\Omega)$, whose elements vanish at $x_1=\pm 1$ and have the period 2π for $y\in Q$. $H^1_{0,p}(\Omega)$ denotes the closure of $C^\infty_{0,p}(\Omega)$ in $H^1(\Omega)$. Note that the solution of (1.1) satisfies the energy conservation

$$\|U(t)\|^2+2\nu\int_0^t |U(t')|_1^2\,dt'=\|U_0\|^2+2\int_0^t (U(t'),f(t'))dt'.$$

One of the both important hands for approximating solutions is an appropriate choice of two discrete spaces for the speed and the pressure. Another is suitable to simulate the conservation.

2. The Scheme

Let M and N be positive integers. Suppose that there exist positive constants d_1 and d_2 such that

$$d_1N\le M\le d_2N.$$

$\mathcal{P}_M(I)$ denotes the space of all polynomials with degree $\le M$. Define

$$V_M=\{v(x_1)\in\mathcal{P}_M(I)/v(-1)=v(1)=0\}.$$

Let $l=(l_2,\cdots,l_n)$, l_q being integers. Set $|l|_\infty=\max\limits_{2\le q\le n}|l_q|$, $|l|=(l_2^2+\cdots+l_n^2)^{\frac12}$, $ly=l_2x_2+\cdots+l_nx_n$ and

$$\tilde V_N=\text{ Span}\{e^{ily}/\,|l|_\infty\le N\}.$$

Let V_N be the subset of $\tilde V_N$, containing all real-valued functions. Define

$$V_{M,N}=(V_M\times V_N)^n,\quad S_{M-1,N}=\{v\in\mathcal{P}_{M-1}(I)\times V_N/\mu(v)=0\}.$$

Let $P^1_{M,N}:(H^1_{0,p}(\Omega))^n\longrightarrow V_{M,N}$ be the projection operator such that for any $u\in(H^1_{0,p}(\Omega))^n$,

$$(\nabla(u-P^1_{M,N}u),\nabla v)=0,\quad\forall v\in V_{M,N}.$$

While $P_{M-1,N}:L^2(\Omega)\longrightarrow\mathcal{P}_{M-1}(I)\times V_N$ is the orthogonal projection such that for any $u\in L^2(\Omega)$,

$$(u-P_{M-1,N}u,v)=0,\quad\forall v\in\mathcal{P}_{M-1}(I)\times V_N.$$

Obviously, if $u\in L^2(\Omega)$ and $\mu(u)=0$, then $\mu(P_{M-1,N}u)=0$.

For continuity equation, we use small parameter technique. Then the incompressible condition is approximated by

$$\beta\frac{\partial P}{\partial t}+\nabla\cdot U=0,\quad\beta>0.$$

To approximate the nonlinear term, we define

$$d(u,v)=\frac12\sum_{j=1}^n v^{(j)}\frac{\partial u}{\partial x_j}+\frac12\sum_{j=1}^n\frac{\partial}{\partial x_j}(v^{(j)}u)$$