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ON MULTIGRID METHODS FOR PARABOLIC PROBLEMS[*1)]

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Abstract

Multigrid methods with nested subspaces and inherited forms are analyzed in an abstract framework that permits application to linear systems of the type that have to be solved at each time level in time-stepping methods for finite element approximations of parabolic problems. Convergence rates that are independent of the space and time steps are obtained in an appropriate time step dependent norm.

1. Introduction

In this article we discuss the solution of linear systems of equations

$$Au = f \tag{1.1}$$

by iterative methods of multigrid type. We are particularly interested in equations of the kind that arise when a parabolic problem, such as

$$U_t(x,t) - \Delta U(x,t) = g(x,t), \quad x \in \Omega,\ t > 0, \tag{1.2}$$

together with initial and boundary conditions, is discretized with respect to the time variable by a time-stepping method, and with respect to the spatial variable by a finite element method. The operator A is then typically of the form $A = zI - k\Delta_h$, where z is a complex number with $\operatorname{Re} z > 0$, k is a small positive parameter (the local time step) and Δ_h is a discrete version of the Laplacian generated by a finite element method with spatial mesh size h. For instance, if the backward Euler method with time step k is used, then (1.2) is first replaced by

$$(U_n - U_{n-1})/k - \Delta U_n = g_n, \quad U_n \approx U(nk),$$

or

$$(I - k\Delta)U_n = U_{n-1} + kg_n,$$

and the finite element discretization of this elliptic problem has the form (1.1) with $A = I - k\Delta_h$. Analogous equations are obtained in connection with other time-stepping methods such as A-stable onestep or multistep methods, see Section 3 below.

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We first formulate iterative methods of multigrid type for solving (1.1), and demonstrate convergence results within an abstract framework that permits application to the situation described above, where the special feature is the presence of the small parameter k. The framework is essentially that used in [3], where applications to elliptic problems are analyzed under weak assumptions, and our results and their proofs are close to those of earlier work, e.g., [1], [2], [4], and [7]. We restrict our discussion here to the case of nested subspaces and inherited forms, and we make regularity assumptions that are satisfied for convex polygonal domains Ω. This makes it possible to organize the theory in straightforward and compact manner, basing on three simple assumptions, and to make our paper selfcontained.

Our convergence results for parabolic problems are expressed in a certain k-dependent energy norm and show rates of convergence that are uniform with respect to h and k. They are of the form required in the analysis of incomplete iterations in [10] and [5]. By combining our results with those of [10] and [5] one may obtain estimates of the total error caused both by the discretization and the iterative solution of the algebraic equations.

Various issues concerning multigrid methods for parabolic problems have been addressed in earlier work, for example, in [1], [8], [9], [12], [13], [14] [16], but in most cases (except [1] and [16]) the convergence analysis is restricted to model problems with a uniform mesh, where Fourier methods can be applied.

2. Abstract Multigrid Analysis

In the first subsection we define the multigrid algorithm and prove some convergence results in the context of symmetric equations in an abstract framework. In the second subsection we extend the analysis to a non-symmetric equation with a special structure.

2.1. Symmetric equations. Let M be a finite dimensional Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\| = (\cdot,\cdot)^{1/2}$ and let $A(\cdot,\cdot)$ be a symmetric, positive definite bilinear form on M. With the linear operator $A : M \to M$ defined by

$$(Au, v) = A(u, v), \quad \forall u, v \in M, \tag{2.1}$$

our concern is to solve the equation

$$Au = f, \quad \text{ for } f \in M. \tag{2.2}$$

Our multigrid method for (2.2) is then the iterative method

$$u^l = u^{l-1} - B(Au^{l-1} - f), \quad l = 1, 2, \dots, \tag{2.3}$$

where $B : M \to M$ is defined as follows. We assume that we are given a nested sequence of subspaces $M_1 \subset \dots \subset M_j \subset M_{j+1} \subset \dots \subset M_J = M$, and define the local version $A_j : M_j \to M_j$ of A by

$$(A_j u, v) = A(u, v), \quad \forall u, v \in M_j. \tag{2.4}$$