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CONSTRUCTION OF A THREE-STAGE DIFFERENCE SCHEME FOR ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, we construct a three-stage difference scheme of 4th order for ordinary differential equations by the method of composing 2nd order schemes symmetrically.

1. Introduction

We know that the difference scheme $Z_{k+1} = Z_k + \frac{h}{2}(f(Z_k)+f(Z_{k+1}))$ with h the step length, is of order two for ordinary differential equations $Z' = f(Z)$, where $Z = Z(t)$. We hope that the three-stage method of the form

$$\begin{cases} Z_1 = Z_0 + c_1 h(f(Z_0) + f(Z_1)) \\ Z_2 = Z_1 + c_2 h(f(Z_1) + f(Z_2)) \\ Z_3 = Z_2 + c_3 h(f(Z_2) + f(Z_3)) \end{cases} \tag{1}$$

would be of order 4 (i.e., $Z_3 - Z(t+h) = O(h^5)$, $Z(t+h)$ is the exact solution at $t+h$ and Z_3 the numerical one) when the parameters c_1, c_2 and c_3 are chosen properly.

We will use the method of Taylor expansion to deal with the simple case when there is only one ordinary differential equation(ODE). When we deal with the case of systems of ODE's, the Taylor expansions become very complex, although it surely can be applied and the same conclusion as in the former case can be got. We introduce another method[2] known as "trees and elementary differentials" to deal with the latter case. In fact, the essence of the two methods are the same, they are just two different ways of expression.

2. Construction for Single Equation

In this section, without specific statements, the values of all functions are calculated at Z_0, and we consider only the terms up to $o(h^4)$ in the following calculations, the higher order terms of h are omitted.

First we calculate the Taylor expansion of the exact solution. Since

$$\dot{Z} = f,\ \ddot{Z} = f'\dot{Z} = f'f,\ Z^{(3)} = f''f^2 + f'^2 f,\ Z^{(4)} = f'''f^3 + 4f''f'f^2 + f'^3 f, \tag{2}$$

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we have, with $Z_0 = Z(t)$,

$$Z(t+h) = Z_0+hf+\frac{h^2}{2!}f'f+\frac{h^3}{3!}(f''f^2+f'^2f)+\frac{h^4}{4!}(f'''f^3+4f''f'f^2+f'^3f)+O(h^5). \quad (3)$$

Now we turn to the Taylor expansion of the numerical solution. We can rewrite (3) as

$$Z_3 = Z_0 + h[c_1 f + (c_1 + c_2)f_1 + (c_2 + c_3)f_2 + c_3 f_3], \quad (4)$$

where for simplicity, we denote $f_i = f(Z_i),\ i = 1, 2, 3$. We need figure out the Taylor expansions of f_1, f_2, f_3. Noticing (4), we just have to expand them up to the terms of order 3 of h.

$$f_i = f + f'(Z_i - Z_0) + \frac{f''}{2!}(Z_i - Z_0)^2 + \frac{f'''}{3!}(Z_i - Z_0)^3 + O(h^4). \quad (5)$$

Since $Z_1 = Z_0 + c_1 h(f_1 + f)$, we then have

$$\begin{aligned} f_1 =& f + (c_1h)2f'f + (c_1h)^2(2f'^2f + 2f''f^2) \\ &+ (c_1h)^3(2f'^3f + 6f''f'f^2 + 4/3f'''f^3) + O(h^4). \end{aligned} \quad (6)$$

We use the same technique to expand the Taylor expansions of f_2, f_3. Since $Z_2 - Z_0 = c_1h(f_1 + f) + c_2h(f_2 + f_1) = c_1hf + (c_1 + c_2)hf_1 + c_2hf_2$, we have

$$\begin{aligned} f_2 =& f + h[2(c_1 + c_2)f'f] + h^2[(c_1 + c_2)^2[2f'^2f + 2f''f^2]] \\ &+ h^3[(c_1 + c_2)(c_1^2 + c_1c_2 + c_2^2)2f'^3f + [(c_1 + c_2)2c_1^2 \\ &+ 2c_2(c_1 + c_2)^2 + 4(c_1 + c_2)^3]f''f'f^2 + 4/3(c_1 + c_2)^3f'''f^3] + O(h^4). \end{aligned} \quad (8)$$

Similarly, we can get

$$\begin{aligned} f_3 =& f + h[2(c_1 + c_2 + c_3)f'f] + h^2[(c_1 + c_2 + c_3)^2 2f'^2f + (c_1 + c_2 + c_3)^2 2f''f^2] \\ &+ h^3[[(c_1 + c_2)c_1^2 + (c_2 + c_3)(c_1 + c_2)^2 + c_3(c_1 + c_2 + c_3)^2]2f'^3f \\ &+ [(c_1 + c_2)c_1^2 + (c_2 + c_3)(c_1 + c_2)^2 + c_3(c_1 + c_2 + c_3)^2 + 2(c_1 + c_2 + c_3)^3]2f''f'f^2 \\ &+ 4/3(c_2 + c_2 + c_3)^3f'''f^3] + O(h^4). \end{aligned} \quad (9)$$

Inserting the Taylor expansions of $f_i (i = 1, 2, 3)$ into (4), we get the Taylor expansion of the numerical solution

$$\begin{aligned} Z_3 =& Z_0 + [c_1 + (c_1 + c_2) + (c_2 + c_3) + c_3]hf \\ &+ [(c_1 + c_2)2c_1 + (c_2 + c_3)2(c_1 + c_2) + c_3 2(c_1 + c_2 + c_3)]h^2f'f \\ &+ [(c_1 + c_2)2c_1^2 + (c_2 + c_3)2(c_1 + c_2)^2 + c_3 2(c_1 + c_2 + c_3)^2]h^3(f''f^2 + f'^2f) \\ &+ [(c_1 + c_2)4/3c_1^3 + (c_2 + c_3)4/3(c_1 + c_2)^3 + c_3 4/3(c_1 + c_2 + c_3)^3]h^4f'''f^3 \\ &+ [(c_1 + c_2)2c_1^3 + (c_1 + c_2)2(c_1^2 + c_2^2 + c_1c_2)(c_2 + c_3) \\ &+ c_3 2[(c_1 + c_2)c_1^2 + (c_2 + c_3)(c_1 + c_2)^2 + c_3(c_1 + c_2 + c_3)^2]]h^4f'^3f \\ &+ [(c_1 + c_2)6c_1^3 + (c_2 + c_3)[4(c_1 + c_2)^3 + 2c_1^2(c_1 + c_2) + 2c_2(c_1 + c_2)^2) + c_3 2(c_1^2(c_1 + c_2) \\ &+ (c_2 + c_3)(c_1 + c_2)^2 + c_3(c_1 + c_2 + c_3)^2 + 2(c_1 + c_2 + c_3)^3]]h^4f''f'f^2 + O(h^5). \end{aligned}$$