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A GALERKIN/LEAST-SQUARE FINITE ELEMENT APPROXIMATION OF BRANCHES OF NONSINGULAR SOLUTIONS OF THE STATIONARY NAVIER-STOKES EQUATIONS *1)

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Abstract

In the author's previous paper [13], a Galerkin/Least-Square type finite element method was proposed and analyzed for the stationary N-S equations. The method is consistent and stable for any combination of discrete velocity and pressure spaces(without requiring the Babuska-Brezzi stability condition). Under the condition that the solution of N-S equations is unique (i.e. in the case of sufficient viscosity or small data), the existence, uniqueness and convergence (at optimal rate) of discrete solution were proved. In this paper, we further investigate the established Galerkin/Least-Square finite element method for the stationary N-S equations. By applying and extending the results of Lopez-Marcos & Sanz-Serna [15], an existence theorem and error estimates are proved in the case of branches of nonsingular solutions.

1. Introduction

For mixed finite element methods solving the stationary (Navier-) Stokes equations in the primitive variables, it is an important convergence stability condition that the Babuska-Brezzi inequality (or inf-sup, or LBB condition) holds for the combination of finite element subspaces[1]. Employment of combinations which fail to satisfy the compatibility condition may yield undesirable pathologies in the approximation of pressure and velocity. Recently, to deal with this potential shortcoming, the so-called CBB[6] or stabilized finite element methods[4], which circumvents the need to satisfy the LBB condition by modifying the variational equations carefully, have been developed under the motivation of SD (or SUPG) methods[7,8]. In addition to works [2-6] on the Stokes problems, a penalty SD type method and a Galerkin/Least-Square method have already been proposed for the stationary Navier-Stokes equations in [11] and [13], respectively. The two methods are stable and different from the method in [12]. So far they have only been analyzed under the condition of unique solution (i.e. in the case of sufficient

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viscosity or small data). Although G. Lube[19] extended the analysis of SD method to quasilinear elliptic problems of second order in the case of branches of nonsingualr solutions following the abstract approach in [1,10], it seems difficult to apply his method to analyze the two stabilized methods[11,13] for the stationary Naver-Stokes equations in the case of high Reynolds number.

In this paper, we further investigate the Galerkin/Least-Square method established in [13] for the stationary N-S equations. By applying and extending the abstract results of Lopez-Marcos & Sanz-Serna[15], the existence, uniqueness and error estimates are proved in the case of branches of nonsingular solutions. It is worth mentioning that the penalty SD type method[11] can be similarly analyzed in the case of branches of nonsingular solutions by our discussions.

For SD method applied to the nonstationary Navier-Stokes equations we referred to papers [8,9,18].

An outline of the paper follows. In Section 2, we introduce some notations which are important for the following presentation. In Section 3, we present the Galerkin/Least-Square method established in [13]. The main result of existence and convergence of branches of discrete solutions contained in Section 4.

2. Notations and Preliminaries

Throughout this paper, Ω is supposed to be a bounded domain in $R^n, n = 2$ or 3, with a Lipschitz continuous boundary Γ. For a scalar function w on a measurable subset $G \subseteq \Omega$, let $\| w \|_{k,p,G}$ and $|w|_{k,p,G}$ be the usual norm and seminorm on the Sobolev space $W^{k,p}(G)$, respectively. For vectorvalued functions $u = (u_1, \cdots, u_n) \in W^{k,p}(G)^n$ and $v = (v_1, \cdots, v_n) \in L^\infty(G)$ we use the following norms and seminorms, respectively.

$$\|u\|^p_{k,p,G} = \sum_{i=1}^n \|u_i\|^p_{k,p,G}, |u|^p_{k,p,G} = \sum_{i=1}^n |u_i|^p_{k,p,G},$$
$$\|v\|_{0,\infty,G} = \max_i \|v_i\|_{0,\infty,G}.$$

$(\cdot,\cdot)_G$ denotes the inner product in $L^2(G)$ and $L^2(G)^n, G \subseteq \Omega$ respectively. In the case of $G = \Omega$ and $p = 2$ we omit the index G and p. Henceforth, we denote by C a generic constant independent of h. Other notations without being specially explained are used in the usual meaning.

In this paper, we consider the following stationary Navier-Stokes equations with boundary conditions.

$$\begin{cases} -\nu\Delta u + u\cdot\nabla u + \nabla p &= f \text{ in } \Omega., \\ \text{div} u &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= 0, \end{cases} \tag{2.1}$$

where $u = (u_1, \cdots, u_n)$ is velocity vector, p the pressure, $f = (f_1, \cdots, f_n)$ the body force, ν the constant inverse Reynolds number. Problem (2.1) is equivalent to the