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HIGH RESOLUTION SCHEMES AND DISCRETE ENTROPY CONDITIONS FOR 2-D LINEAR CONSERVATION LAWS[*,1)]

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Abstract

In this paper, fully discrete entropy conditions of a class of high resolution schemes with the MmB property are discussed by using the theory of proper discrete entropy flux for the linear scalar conservation laws in two dimensions. The theoretical resluts show that the high resolution schemes satisfying fully discrete entropy conditions with proper discrete entropy flux cannot preserve second order accuracy in the case of two dimensions.

1. Introduction

Consider 2-D hyperbolic conservation laws:

$$\begin{aligned} &\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}+\frac{\partial g(u)}{\partial y}=0\ , \\ &u(x,y,0)=u_0(x,y)\ . \end{aligned} \tag{1.1}$$

The research of numerical methods for the equations has been developed rapidly in this decade. Since appearance of the concept of TVD(total variation diminishing) schemes, various high resolution schemes (TVD,TVB (total variation bounded[6]), ENO (essentially non-oscillatory[2]), MmB (Maxima minima Bounded preserving[10]) schemes etc.) have been applied successfully to computational fluid dynamics. Recently, the convergence of difference schemes by using every ways are discussed. The convergence of numerical methods for hyperbolic conservation laws depends on the entropy condition and some kinds of stability of difference solutions such as the total variation stability. However, there exists some relationship between the entropy condition and nonlinear stability of numerical solutions. Previously constructing difference schemes always based on some kinds of total variation stability (TVD, TVB, ENO, and MmB etc.). Then these schemes are modified so that the entropy condition can be satisfied. Some quantities depending on the grid width are often introduced when these modifications are made. Generally, the difference schemes only depend on the grid ratio

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but independ of the grid width. So, it is meaningful to construct schemes satisfying the entropy condition without intruducing the quantities depending on the grid width. M. Merriam[3] and T. Sonar[8] put out the concept of the proper discrete entropy flux. That is discretizating the entropy flux by using the proper way so that the entropy condition can be satisfied and simultaneously the difference solution satisfies some kind of total variation stability. In [11], N. Zhao and H. Wu discussed the relationship between entropy conditions and nonlinear stability for 1-D scalar linear conservation laws, and obtained second order accurate TVD schemes using limiters. Based on the similar procedure, in this paper, we discuss the relationship between the discrete entropy conditions and the MmB property in the case of two dimensions. Unfortunately, the theoretical results show that a class of high resolution schemes satisfying the discrete entropy condition with the proper discrete entropy flux cannot preserve second order accuracy for linear scalar hyperbolic conservation laws in two dimensions.

2. MmB Schemes in Two Dimensions

In this section, let us review the MmB schemes in two dimensions introduced by H. Wu and S. Yang in [10].

Consider the difference schemes for 2-D scalar equations

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0 \ , \ a > 0 \ , \ b > 0 \ , \tag{2.1}$$

where a and b are constants. Let $\lambda = a\Delta t/\Delta x$, $\mu = b\Delta t/\Delta y \geq 0$, be the Courant numbers, and $u^n_{j,k}$ the approximating function value of the solution at the mesh point (x_j, y_k, t^n).

In general, we have the following partially 'upwind' second order accurate scheme to approximate the equation (2.1) (the notations are conventional, $u_{j,k} = u^n_{j,k}$)

$$\begin{aligned} u^{n+1}_{j,k} =& u_{j,k} - \lambda(u_{j,k} - u_{j-1,k}) - \frac{\lambda(1-\lambda)}{2}(u_{j+1,k} - 2u_{j,k} + u_{j-1,k}) \\ &- \mu(u_{j,k} - u_{j,k-1}) - \frac{\mu(1-\mu)}{2}(u_{j,k+1} - 2u_{j,k} + u_{j,k-1}) \\ &+ \lambda\mu\left((u_{j,k} - u_{j-1,k}) - (u_{j,k-1} - u_{j-1,k-1})\right). \end{aligned} \tag{2.2}$$

The scheme (2.2) is not MmB, it may cause oscillations for non-smooth solutions. So, H. Wu and S. Yang constructed the following flux limited version of the modification of (2.2) in [10]

$$\begin{aligned} u^{n+1}_{j,k} =& u_{j,k} - \lambda\Delta_{j-\frac{1}{2},k}u - \frac{\lambda(1-\lambda)}{2}\left(\varphi_{j,k}\Delta_{j+\frac{1}{2},k}u - \varphi_{j-1,k}\Delta_{j-\frac{1}{2},k}u\right) \\ &+ \frac{\lambda\mu}{2}\left[\Theta_{j,k-\frac{1}{2}}\Delta_{j,k-\frac{1}{2}}u - \Theta_{j-1,k-\frac{1}{2}}\Delta_{j-1,k-\frac{1}{2}}u\right] \\ &- \mu\Delta_{j,k-\frac{1}{2}}u - \frac{\mu(1-\mu)}{2}\left(\psi_{j,k}\Delta_{j,k+\frac{1}{2}}u - \psi_{j,k-1}\Delta_{j,k-\frac{1}{2}}u\right) \end{aligned}$$