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A MULTI-GRID ALGORITHM FOR STOKES PROBLEM*1)

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Abstract

In this paper we describe a multi-grid algorithm for the penalty procedure of Stokes problem. It is proved that the convergence rate of the algorithm is bounded away from 1 independently of the meshsize. For convenience, we only discuss Jacobi relaxation as smoothing operator in detail.

1. Introduction

Consider the Stokes problem

$$\begin{cases} -\mu \triangle \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^d, $d = 2$ or 3. Since, within a code for the numerical solution of the Navier-Stokes equations, one needs an efficient Stokes-solver, the multigrid method is very attractive for the solution of the discrete analogue of (1.1).

Brezzi and Douglas[6] have applied a penalty procedure for (1.1) with the C^0-piecewise linear element of velocity and pressure and achieved an optimal convergence rate. In this paper we establish a multi-grid algorithm for the penalty procedure of Stokes problem and show that the convergence rate of the algorithm is bounded away from 1 independently of the meshsize.

The general structure of our convergence analysis for the multi-grid algorithm is similar to that of Bank and Dupont[2,3] and Hackbusch[8]. The smoothing properties are given in terms of a mesh-dependent norm. The approximation properties are obtained from error estimates in terms of Sobolev spaces. The connection between the associated scales of Sobolev spaces, however, requires some special considerations. It is performed via the duality technique of Aubin-Nitsche. To simplify the analysis we only consider Jacobi relaxation as smoothing procedure in detail.

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2. A Multi-Grid Algorithm

A mixed formulation of (1.1) is given by the finding of $[\mathbf{u},p] \in \mathbf{H}_0^1(\Omega)\times \hat{L}^2(\Omega)$ such that

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}) & \forall \mathbf{v}\in \mathbf{H}_0^1(\Omega), \\ b(\mathbf{u},q) \qquad\qquad = 0 & \forall q \in \hat{L}^2(\Omega) \end{cases} \tag{2.1}$$

with the bilinear form

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= \mu \sum_{i=1}^{k} (\bigtriangledown u_i, \bigtriangledown v_i) = \mu \sum_{i,j=1}^{k} (\frac{\partial\, u_i}{\partial\, x_j}, \frac{\partial\, v_i}{\partial\, x_j}), \\ b(\mathbf{v},q) &= -(\mathrm{div}\ \mathbf{v}, q) \end{aligned}$$

on $\mathbf{H}_0^1(\Omega)\times\mathbf{H}_0^1(\Omega)$, $\mathbf{H}_0^1(\Omega)\times\hat{L}^1(\Omega)$. Here, $(\cdot,\cdot)$ is the inner product in L^2. Moreover, $H^k(\Omega)$, $k\in N$, and $L^2(\Omega) = H^0(\Omega)$ are the usual Sobolev and Lebesgue spaces equipped with the norms[1]

$$\|u\|_k = \{\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u(x)|^2\ dx\}^{\frac{1}{2}}.$$

Furthermore, $\mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^2$. We use a circumflex " $\hat{}$ " above a function space to denote the subspace of the elements with mean value zero.

Let T_0 be a partition of Ω into d-simplices and h_0 be the longest side of the simplices of T_0. We suppose that the simplices of T_0 satisfy the usual regularity assumptions for finite elements[7] and that

$$\begin{aligned} h_K &\le c_0\rho_K, \qquad \forall K\in T_0, \\ h_K &:= \mathrm{diam}(K), \\ \rho_K &:= \sup\{\mathrm{diam}(B) \mid B \text{ is a ball contained in } K\}, \end{aligned} \tag{2.2}$$

where c_0 is not large. The partitions T_k, $1\le k\le R$, are defined by dividing each $K\in T_{k-1}$ into 2^d d-simplices by joining the midpoints of the sides (cf. Fig. 1). Then $h_k = 2^{-k}h_0$, and the partitions T_k satisfy the regularity assumption (2.2) with the same constant c_0.

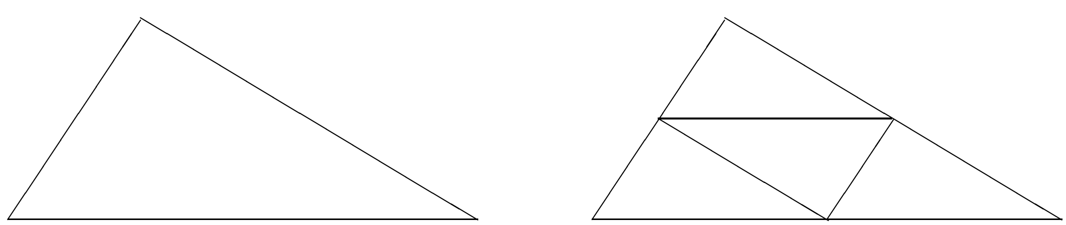

Fig. 1. Subdivision of triangles in the construction of T_k from T_{k-1}