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A LINEARIZED DIFFERENCE SCHEME FOR THE KURAMOTO-TSUZUKI EQUATION*1)

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Abstract

In this paper, a linearized three-level difference scheme is derived for the mixed boundary value problem of Kuramoto-Tsuzuki equation, which can be solved by double-sweep method. It is proved that the scheme is uniquely solvable and second order convergent in energy norm.

1. Introduction

Tsertsadze[1] studied the finite difference method for the mixed boundary value problem of Kuramoto-Tsuzuki equation

$$\frac{\partial w}{\partial t} = (1+ic_1)\frac{\partial^2 w}{\partial x^2} + w - (1+ic_2)\left|w\right|^2 w, \quad 0<x<1,\, 0<t\le T \tag{1.1}$$

$$\frac{\partial w}{\partial x}(0,t)=0, \qquad \frac{\partial w}{\partial x}(1,t)=0, \quad 0<t\le T \tag{1.2}$$

$$w(x,0)=w_0(x), \quad 0\le x\le 1 \tag{1.3}$$

where c_1 and c_2 are real constants, $w(x,t)$ and $w_0(x)$ complex valued functions. Divide $[0,1]$ into$M$ subintervals and $[0,T]$ into K subintervals with meshsizes h and τ respectively. Tsertsadze[1] constructed for (1.1)-(1.3) the following difference scheme

$$\delta_t w_0^{k+\frac12} = (1+ic_1)\frac{2}{h^2}(w_1^{k+\frac12} - w_0^{k+\frac12}) + w_0^{k+\frac12} - (1+ic_2)\left|w_0^{k+\frac12}\right|^2 w_0^{k+\frac12},$$
$$0\le k\le K-1 \tag{2.1}$$

$$\delta_t w_j^{k+\frac12} = (1+ic_1){\delta_x}^2 w_j^{k+\frac12} + w_j^{k+\frac12} - (1+ic_2)\left|w_j^{k+\frac12}\right|^2 w_j^{k+\frac12},$$
$$1\le j\le M-1,\, 0\le k\le K-1 \tag{2.2}$$

$$\delta_t w_M^{k+\frac12} = (1+ic_1)\frac{2}{h^2}(w_{M-1}^{k+\frac12} - w_M^{k+\frac12}) + w_M^{k+\frac12} - (1+ic_2)\left|w_M^{k+\frac12}\right|^2 w_M^{k+\frac12},$$
$$0\le k\le K-1 \tag{2.3}$$

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$$w_j^0 = w_0(x_j), \qquad 0 \le j \le M \tag{2.4}$$

where $x_j = jh, t_k = k\tau, w_j^k$ the approximation of $w(x_j, t_k)$, $w_j^{k+\frac{1}{2}} = (w_j^{k+1} + w_j^k)/2$, $\delta_t w_j^{k+\frac{1}{2}} = (w_j^{k+1} - w_j^k)/\tau$, $\delta_x^2 w_j^k = (w_{j+1}^k - 2w_j^k + w_{j-1}^k)/h^2$ and proved that the difference scheme is convergent in energy norm with the convergence rate of order $O(h^{3/2})$ when $\tau = O(h^{2+\epsilon})$ $(\epsilon > 0)$. (2) is nonlinear.

In this paper, for generality, we consider inhomogeneous equation. In other words, instead of (1.1), we consider

$$\frac{\partial w}{\partial t} = (1 + ic_1)\frac{\partial^2 w}{\partial x^2} + w - (1 + ic_2)\left|w\right|^2 w + f(x,t), \quad 0 < x < 1, 0 < t \le T \tag{1.1$'$}$$

where $f(x,t)$ is a known complex valued smooth function. We develop for (1.1') and (1.2)-(1.3) the difference scheme

$$\Delta_t w_0^k = (1 + ic_1)\frac{2}{h^2}(w_1^{\hat{k}} - w_0^{\hat{k}}) + w_0^{\hat{k}} - (1 + ic_2)\left|w_0^k\right|^2 w_0^{\hat{k}} + f(\frac{h}{3}, t_k),$$
$$1 \le k \le K - 1 \tag{3.1}$$

$$\Delta_t w_j^k = (1 + ic_1){\delta_x}^2 w_j^{\hat{k}} + w_j^{\hat{k}} - (1 + ic_2)\left|w_j^k\right|^2 w_j^{\hat{k}} + f(x_j, t_k),$$
$$1 \le j \le M - 1, 1 \le k \le K - 1 \tag{3.2}$$

$$\Delta_t w_M^k = (1 + ic_1)\frac{2}{h^2}(w_{M-1}^{\hat{k}} - w_M^{\hat{k}}) + w_M^{\hat{k}} - (1 + ic_2)\left|w_M^k\right|^2 w_M^{\hat{k}} + f(1 - \frac{h}{3}, t_k),$$
$$1 \le k \le K - 1 \tag{3.3}$$

$$w_j^0 = w_0(x_j), \quad w_j^1 = w_0(x_j) + \tau w_1(x_j), \quad 0 \le j \le M \tag{3.4}$$

where

$$w_1(x) = (1 + ic_1)\frac{d^2 w_0(x)}{dx^2} + w_0(x) - (1 + ic_2)\left|w_0(x)\right|^2 w_0(x) + f(x, 0)$$
$$w_j^{\hat{k}} = (w_j^{k+1} + w_j^{k-1})/2, \qquad \Delta_t w_j^k = (w_j^{k+1} - w_j^{k-1})/(2\tau).$$

The scheme (3) is a tridiagonal system of linear algebraic equations, which can be solved by double-sweep method. We suppose $\tau = \alpha h^{\frac{1}{4}+\epsilon}$, where α and ϵ are any two positive constants. In next two sections, we will prove that (3) is uniquely solvable and convergent in energy norm with convergence rate of order $O(\tau^2 + h^2)$. Farthermore, we will see that the optimal choice is $\epsilon = 3/4$ or $\tau = O(h)$.

Let $u \equiv \{u_j\}_{j=0}^M$ be a net function on $I \equiv \{x_j\}_{j=0}^M$, define the L_2 norm

$$\|u\| = \sqrt{h\left(\frac{1}{2}{u_0}^2 + \sum_{j=1}^{M-1}{u_j}^2 + \frac{1}{2}{u_M}^2\right)}.$$

2. Solvability

Theorem 1. *The difference scheme* (3) *is uniquely solvable.*