Journal of Computational Mathematics, Vol.14, No.1, 1995, 71–97.

DISCRETE APPROXIMATIONS FOR SINGULARLY PERTURBED BOUNDARY VALUE PEOBLEMS WITH PARABOLIC LAYERS, I[*1)]

P.A. Farrell
(*Department of Mathematics and Computer Science, Kent State University, USA*)
P.W. Hemker
(*CWI, Center for Mathematics and Computer Science, Amsterdam, The Netherlands*)
G.I. Shishkin
(*IMM, Institute of Mathematics and Mechanics, Ural Branch of the Russian Academy of Science, Ekaterinburg, Russia*)

Abstract

In this series of three papers we study singularly perturbed (SP) boundary value problems for equations of elliptic and parabolic type. For small values of the perturbation parameter parabolic boundary and interior layers appear in these problems. If classical discretisation methods are used, the solution of the finite difference scheme and the approximation of the diffusive flux do not converge uniformly with respect to this parameter. Using the method of special, adapted grids, we can construct difference schemes that allow approximation of the solution and the normalised diffusive flux uniformly with respect to the small parameter.

We also consider singularly perturbed boundary value problems for convection-diffusion equations. Also for these problems we construct special finite difference schemes, the solution of which converges ε-uniformly. We study what problems appear, when classical schemes are used for the approximation of the spatial derivatives. We compare the results with those obtained by the adapted approach. Results of numerical experiments are discussed.

In the three papers we first give an introduction on the general problem, and then we consider respectively (i) Problems for SP parabolic equations, for which the solution and the normalised diffusive fluxes are required; (ii) Problems for SP elliptic equations with boundary conditions of Dirichlet, Neumann and Robin type; (iii) Problems for SP parabolic equation with discontinuous boundary conditions.

General Introduction

Consider a substance (or admixture) in a solution with a flux satisfying Fick's law, and with distribution given by a diffusion equation. Let the initial concentration of

[*] Received March 4, 1995.
[1)] This work was supported by NWO through grant IB 07-30-012.

the admixture in the material as well as the concentration of the admixture on the boundary of the body be known. It is required to find the distribution of admixture in the material at any given time and also the quantity of admixture (that is the diffusive flux) emitted from the boundaries into the exterior environment. Such problems are of interest in environmental sciences in determining the pollution entering the environment from manufactured sources, such as houses, factories and vehicles, and from industrial and agricultural waste disposal sites, and also in chemical kinetics where the chemical reactions are described by reaction-diffusion equations.

In considering such problems, it is important to note that the diffusion Fourier number, which is given by the diffusion coefficient of the admixture in materials, can be sufficiently small that large variations of concentration occur along the depth of the material. For small values of the Fourier number, diffusion boundary layers appear. Therefore these problems exhibit a singularly perturbed character. The mathematical formulation of such problems have a perturbation parameter which is a small coefficient (the diffusion Fourier number) multiplying the highest derivatives of the differential equation.

Even in the case where only the approximate solution of the singularly perturbed boundary value problem is required, classical numerical methods, such as finite difference schemes and finite element methods[15, 16, 17] exhibit unsatisfactory behaviour. This arises because the accuracy of the approximate solution depends inversely on the perturbation parameter value and thus it deteriorates as the parameter decreases. In [18] it was shown that the use of classical numerical methods does not give approximate solutions with acceptable accuracy even for very fine grids. Thus, even the use of computers with extremely large capacity will not guarantee acceptable accuracy in the answer. To be more precise, it can be shown that the error in the approximate solution on any arbitrarily fine grid is greater than some positive number (independent of the number of grid nodes), for a sufficiently small value of the perturbation parameter (the diffusion Fourier number). For some applied problems such solution accuracy can be satisfactory. However even in these cases dissatisfaction can be caused by the lack of a guarantee than the use of a finer grid will increase the accuracy of the approximation.

More serious problems occur when an accurate approximation of the spatial derivatives of the solution is also required. For example, in order to determine the quantity of admixture which enters the environment per unit of time, it is necessary to compute the gradient of the concentration of the substance along the normal to the surface of the material. When classical finite difference schemes are used it can be expected that errors in the computed diffusive flux will be much larger than those of the computed concentration. Such errors in evaluating fluxes can be often of unacceptable magnitude.

Similar difficulties appear also in problems of heat exchange in cases where the heat Fourier number can take any arbitrary small value. One often requires an accurate approximation of the thermal flux on a boundary of the body.

This series of papers is devoted to the construction of numerical approximations,