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ON BOUNDARY TREATMENT FOR THE NUMERICAL SOLUTION OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH FINITE DIFFERENCE METHODS*

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1. Introduction

Consider the incompressible Navier-Stokes equations (INSE)

$$\frac{\partial \mathbf{w}}{\partial t} + u\frac{\partial \mathbf{w}}{\partial x} + v\frac{\partial \mathbf{w}}{\partial y} + \text{grad } p = \alpha \text{ div grad } \mathbf{w} \tag{1}$$

$$\text{div } \mathbf{w} = 0 \tag{2}$$

on region Ω, where $\mathbf{w} = (u, v)'$, with initial condition

$$\mathbf{w}(x, y, 0) = \mathbf{w}^0(x, y) \quad \text{on } \Omega$$

satisfying (2) and boundary conditions satisfying

$$\oint w_n ds = 0 \quad \text{on } \partial\Omega \tag{3}$$

Specific boundary conditions for the INSE and numerical boundary conditions for its numerical solution have been controversial issues in computational fluid dynamics. An attempt is made to clarify some of the problems in this work, based on the author's experience, see Huang et al. [1] for example, and a paper of Perot[2]. In the latter, the issue on numerical boundary conditions is resolved with the linear algebra approach. This approach will be used here on the 'delta' form of the finite difference equation, leading to $O(\Delta t^2)$ results and will be extended to dimensional split and uniform boundary treatment. It is found that no numerical boundary conditions are needed, but numerical boundary conditions similar to those of Kim and Moin[3] and Yanenko[4], for auxiliary velocity and for intermediate velocity with dimensional split respectively, are desirable for uniform boundary treatment.

It is the author's belief that interior and boundary schemes should be developed together such that their properties match as much as possible. This, amongst other

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reasons, leads to a change from the staggered mesh to the half-staggered mesh shown in Fig. 1. On this mesh, there is no half interval differencing near the boundary and pressure boundary condition remains unnecessary, which is mathematically correct as we see in §2 on boundary conditions for the INSE. In §3, we derive as [1] the pressure correction projection method via approximate factorization (AF) as fractional step method with the Crank-Nicolson scheme. We know that INSE, upon spatial discretization, forms a differential algebraic system, and the local second order temporal accuracy of the Crank-Nicolson scheme implies global second order temporal accuracy, see Hairer [5]. Van Kan [6] has shown that this scheme with pressure correction preserves its global accuracy. In §3 and §4, we show that all numerical boundary conditions considered are local second order approximations for the auxiliary velocity, which should not effect the global accuracy. In §5, results of preliminary numerical experiment are presented confirming some of the conclusions.

2. Boundary Conditions for INSE

In this section, we state some ‘proper’ boundary conditions for the INSE. Now, normal mode analysis applied to the ‘frozen coefficient’ systems for small perturbation of hyperbolic systems can lead to significant results. Here, normal mode analysis applied to the corresponding INSE system

$$\begin{gathered}\frac{\partial \dot{\mathbf{w}}}{\partial t} + u\frac{\partial \dot{\mathbf{w}}}{\partial x} + v\frac{\partial \dot{\mathbf{w}}}{\partial y} + \text{grad } \dot{p} - \alpha \text{ div grad } \dot{\mathbf{w}} = 0 \\ \text{div}\dot{\mathbf{w}} = 0\end{gathered} \tag{4}$$

yields only that the number of boundary conditions is two. So we turn to the energy method and obtain

$$\frac{d}{dt}(\dot{\mathbf{w}}, \dot{\mathbf{w}}) \leq -\oint w_n \ \dot{\mathbf{w}}.\dot{\mathbf{w}} ds + 2\alpha \ \oint \dot{\mathbf{w}}.\frac{\partial \dot{\mathbf{w}}}{\partial n} ds - 2 \ \oint \dot{p}\dot{w}_n ds$$

which we set ≤ 0 as sufficient condition for the boundary conditions to be ‘proper’. In the above inequality, $(\mathbf{u}, \mathbf{v}) = \int\int \mathbf{u}.\mathbf{v} dxdy$. On the left boundary, say, setting the integrand to be ≤ 0 everywhere, i.e.

$$u(\dot{u}^2 + \dot{v}^2) + 2\dot{u}\dot{p} - 2\alpha\dot{u}\frac{\partial \dot{u}}{\partial x} - 2\alpha\dot{v}\frac{\partial \dot{v}}{\partial x} \leq 0 \tag{5}$$

we deduce the following ‘proper’ boundary conditions:

$$\begin{array}{rcl} \text{solid wall} & : & \dot{u} = 0 \text{ and } \dot{v} = 0 \ (u \text{ and } v \text{ given}) \\ \text{inflow}(u > 0) & : & \dot{u} = 0 \text{ and } \dot{v} = 0 \ (u \text{ and } v \text{ given}) \\ \text{outflow}(u < 0) & : & \dot{u} = 0 \text{ or } \dot{p} - \alpha\dfrac{\partial \dot{u}}{\partial x} = 0 \ (u \text{ or } p - \alpha\dfrac{\partial u}{\partial x} \text{ given}) \\ & \text{and} & \dot{v} \text{ or } \dfrac{\partial \dot{v}}{\partial x} = 0 \ (v \text{ or } \dfrac{\partial v}{\partial x} \text{ given}) \end{array}$$