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PRECONDITIONED CONJUGATE GRADIENT METHODS FOR INTEGRAL EQUATIONS OF THE SECOND KIND DEFINED ON THE HALF-LINE*[1)]

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Abstract

We consider solving integral equations of the second kind defined on the half-line $[0, \infty)$ by the preconditioned conjugate gradient method. Convergence is known to be slow due to the non-compactness of the associated integral operator. In this paper, we construct two different circulant integral operators to be used as preconditioners for the method to speed up its convergence rate. We prove that if the given integral operator is close to a convolution-type integral operator, then the preconditioned systems will have spectrum clustered around 1 and hence the preconditioned conjugate gradient method will converge superlinearly. Numerical examples are given to illustrate the fast convergence.

1. Introduction

In this paper, we study numerical solutions to integral equations of the second kind defined on the half-line. More precisely, we consider the equation

$$y(t) + \int_0^\infty a(t,s)y(s)ds = g(t), \quad 0 \le < t < \infty \tag{1}$$

where $g(t)$ is a given function in $L_2[0,\infty)$ and the kernel function $a(s,t)$ is in $L_2(R^2)$. One way of solving (1) is by the projection method [3] where the solution $y(t)$ is approximated by the solution $y_\tau(t)$ of the finite-section equation

$$y_\tau(t) + \int_0^\tau a(t,s)y_\tau(t) = g(t), \quad 0 \le t \le \tau. \tag{2}$$

It is shown in [3] that

$$\lim_{\tau\to\infty} \|y_\tau - y\|_{L_p[0,\tau)} = 0, \quad 1 \le p < \infty.$$

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The finite-section equation (2) can be solved numerically by either direct or iterative methods. For a fixed τ, the finite-section operator A_τ defined by

$$(A_\tau x)(t) = \begin{cases} \int_0^\tau a(t,s)x(s)ds, & 0 \le t \le \tau, \\ 0, & t > \tau. \end{cases} \tag{3}$$

is a compact operator. Therefore, the spectrum of the operator $I + A_\tau$ is clustered around 1 and hence solving (2) by iterative methods such as the conjugate gradient (CG) method will be less expensive than direct methods. However, as $\tau \to \infty$, the spectrum of A_τ becomes dense in that of A, where A is defined as

$$Ax(t) = \int_0^\infty a(t,s)x(s)ds, \quad 0 \le t < \infty,$$

and hence the convergence rate of the CG method will deteriorate, see the numerical results in Section 5.

One way of speeding up the convergence rate of the CG method is to apply a preconditioner to (2). Thus instead of solving (2), we solve the preconditioned equation

$$(I + H_\tau)^{-1}(I + A_\tau)y_\tau(t) = (I + H_\tau)^{-1}g(t). \tag{4}$$

We will call the operator H_τ a preconditioner for the operator A_τ. A good preconditioner H_τ is an operator that is close to A_τ in some norm and yet the operator equation

$$(I + H_\tau)x(t) = f(t) \tag{5}$$

is easier to solve than (2) for arbitrary function $f \in L_2[0,\tau]$. A class of candidates is the class of operators of the form

$$H_\tau x(t) = \int_0^\tau h_\tau(t-s)x(s)ds, \quad 0 \le t \le \tau,$$

where the kernel functions h_τ are periodic in $[0,\tau]$. They are called circulant integral operators in [5]. The eigenfunctions and eigenvalues of the operator H_τ are given by

$$u_m(t) = \frac{1}{\sqrt{\tau}} e^{2\pi i m t/\tau}, \quad m \in Z, \tag{6}$$

and

$$\lambda_m = \sqrt{\tau}(h_\tau, u_m)_\tau = \sqrt{\tau}\int_0^\tau h_\tau(t)\bar{u}_m(t)dt, \quad m \in Z, \tag{7}$$

Therefore, (5) can be solved effeciently by using the Fourier transforms.

The convergence rate of solving the preconditioned system (4) with CG method depends on how close the operator $(I+H_\tau)$ is to the operator $(I+A_\tau)$, see Axelsson and Barker [1,p.28]. Therefore, a natural idea is to find the circulant integral operator H_τ that minimizes the difference $A_\tau - H_\tau$ in some norm over all circulant integral operators. In this paper, we will consider the minimization in the Hilbert-Schmidt norm $|||\cdot|||$. We will construct two different kinds of circulant integral preconditioners for A_τ. The first one minimizes $|||A_\tau - H_\tau|||$ and the second one minimizes $|||I - (I + H_\tau)^{-1}(I + A_\tau)|||$.