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SUBCONVERGENCE OF FINITE ELEMENTS AND A SELF–ADAPTIVE ALGORITHM*1)2)

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1. Introduction

So far, there have been many papers concerning with the superconvergence of finite elements. Under some conditions, a higher convergence accuracy is obtained at some specific points. This phenomenon is called superconvergence. The theory of supercovergence[1] tells us that, in order to gain superconvergence, two conditions must be satisfied. One is the good subdivision and the second is the existence of locally sufficient smooth solution. If these conditions are not satisfied, e.g., low smoothness of the solution, occurs a contrary phenomenon that some local errors is greater than that of average errors. This phenomenon, we call *subconvergence.*

Subconvergence is conflicting with superconvergence. It often occurs in the neighourhood of singular points, where higher accuracy algorithm is followed with great interest by engineers but it is very difficult. So it is necessary to study the theory of subconvergence.

Bubuska[2] proposed the h–p version that locally refining subdivision or locally increasing the degree of piecewise polynomials gets locally higher accuracy. To implement this method, it is necessary to use the posterior data to find the elements at which the subconvergence occurs. Otherwise, refining all elements and not distinguishing the 'good' ones with the 'bad' ones, the algorithm is very inefficiency and it is not a high accuracy one.

This paper uses the idea of subconvergence to obtain a simple criterion Δ_e, which only depends on the approximate solution u^h and the function f. Comparing the sizes of Δ_e, the element e_0 with the worst convergence can be found. Then, e_0 and the neighbouring elements are refined by the h-version or p-version. Repeating this process many times, the high accuracy results may be obtained for the problem with singularity.

The outline of this paper is as follows. Section 1 presents some definitions about the subconvergence. Section 2 gives a method of locally refinement or local increasing the degree of polynomials and some main results. In section 3 several examples are provided.

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2) Reported in Tianjin Conference on Numerical Mathematics in 1991.

2. Subconvergence Point, Subconvergence Element and A Self–Adaptive Method

Consider the model problem: to find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad v \in H_0^1(\Omega) \tag{2.1}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, $f \in L^2(\Omega)$, and $a(\cdot,\cdot)$ an elliptic bilinear form with some singularity, Ω a bounded domain in $R^N (N = 1, 2, 3)$. We call

$$\|u\| = \sqrt{a(u, u)}$$

the energy norm.

Let T^h be a regular subdivision on Ω, $S^h(\Omega)$ a piecewise linear finite element space and

$$S_0^h(\Omega) = \{v \in S^h(\Omega) : v|_{\partial\Omega} = 0\}$$

Suppose $u^h \in S_0^h(\Omega)$ be the Galerkin approximation of the solution u of problem (2.1). Then the energy

$$\left\|u - u^h\right\|^2 = \sum_{e \in T^h} \left\|u - u^h\right\|_e^{\ 2}$$

Generally speaking, the element $e \in T^h$ with large $\left\|u - u^h\right\|_e^{\ 2}$ is a *subconvergence element* or bad element.

Now, we want to find a criterion to say $\left\|u - u^h\right\|_e^2$ large or small. To do this, we denote, by $S_0^{h/2}(\Omega)$, $p-$degree finite element space on T^h or the linear finite element space after many times refinement of T^h, and denote, by $\bar{G}$, the set of base functions of finite element space $S_0^{h/2}(\Omega)$. It is obvious that there exists a $\delta_0 \in G_e \equiv \operatorname{span}\{\phi \in \bar{G} : (\,supp\ \phi) \cap e \neq \emptyset\}$ such that

$$\Delta_e \equiv \left| a(u - u^h, \delta_0) \,/\, \|\delta_0\| \right| = \max_{\delta \in G_e} \left| a(u - u^h, \delta) \,/\, \|\delta\| \right| \tag{2.2}$$

If $\Delta_{e_0} = \max_{e \in T^h}\{\Delta_e\}$, we call e_0 the bad element of T^h.

If we refine T^h by mid-point locally again and again and the point z_0 belongs to the bad element infinitly, we call z_0 *subconvergence point.* Since

$$\Delta_e \equiv \left| a(u - u^h, \delta_0) \,/\, \|\delta_0\| \right| \tag{2.3}$$

it is a posterior datum, which can be determined by the computer. It can be seen below that replacing the error energy $\left\|u - u^h\right\|_e$ on element e with Δ_e is reasonable (see Th.1 & Th. 2).

Let $e \in T^h$ be a bad element. Refine e by mid-point and then denote all new nodal points by N_e'. For each point $z \in N_e'$, we have a basis function $\phi_z \in \bar{G}$. Introduce the following notations

$$S_0^{h/2}(D_e) = \operatorname{span}\{\phi_z : z \in N_e'\} \tag{2.4}$$