Journal of Computational Mathematics, Vol.14, No.4, 1996, 345–362.

OPTIMAL INTERIOR AND LOCAL ERROR ESTIMATES OF A RECOVERED GRADIENT OF LINEAR ELEMENTS ON NONUNIFORM TRIANGULATIONS*

I. Hlaváček M. Křížek
(*Mathematical Institute, Žitná 25, CZ-11567, Prague 1, Czech Republic*)

Abstract

We examine a simple averaging formula for the gradient of linear finite elements in R^d whose interpolation order in the L^q-norm is $\mathcal{O}(h^2)$ for $d < 2q$ and nonuniform triangulations. For elliptic problems in R^2 we derive an interior superconvergence for the averaged gradient over quasiuniform triangulations. Local error estimates up to a regular part of the boundary and the effect of numerical integration are also investigated.

1. Introduction

Consider a model elliptic boundary value problem

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) = f, \quad \text{in } \Omega, u = 0, \quad \text{on } \partial\Omega, 1.1$$

where $\Omega \subset R^d$, $d = 1, 2, 3$, is a bounded polyhedral domain with a Lipschitz boundary, $f \in L^2(\Omega)$, a_{ij} are Lipschitz-continuous functions and the matrix $\mathcal{A} = (a_{ij})$ is symmetric and uniformly positive definite with respect to $x \in \Omega$.

It is known that the finite element method applied to (1.1) may produce some superconvergence phenomena even if the used meshes are nonuniform[5,8,9,10,12]. In a recent paper [6], an interior error estimate for the recovered gradient of Galerkin piecewise linear approximations has been proposed in the case $d = 2$. This result, however, has required a high global regularity of the solution of the boundary value problem. In the present paper we derive other error estimates over some subdomains for problems of low regularity. We employ some results of [9, 13] together with a series of modified lemmas of our recent paper [6].

* Received November 9, 1994.

In Section 2 we establish an optimal interior error estimate on subdomains in L^2-norm, whereas Section 3 is devoted to an error estimate in the so-called discrete interior L^2-norm. The effect of numerical integration is treated in Section 4 on a simple example. We present local error estimates up to a regular part of the boundary in Section 5, 6 and 7, in the continuous and discrete L^2-norm, respectively.

Throughout the paper $C, C', \ldots$ are generic positive constants and $\|\cdot\|$ is the Euclidean norm. The symbol $W_q^k(\Omega)$ stands for the Sobolev space equipped with the standard norm $\|\cdot\|_{k,q,\Omega}$ and seminorm $|\cdot|_{k,q,\Omega}$. In particular, we write

$$\|\cdot\|_{k,\Omega} = \|\cdot\|_{k,2,\Omega}, \qquad |\cdot|_{k,\Omega} = |\cdot|_{k,2,\Omega},$$

and $(.,.)_{0,\Omega}$ is the scalar product in $L^2(\Omega)$. The subspace of $W_2^1(\Omega)$ whose functions have vanishing traces is denoted by $\circ \rightarrow W_2^1(\Omega)$. The weak solution $u \in \circ \rightarrow W_2^1(\Omega)$ of (1.1) is defined by the relation

$$a(u,v) = (f,v)_{0,\Omega} \quad \forall v \in \circ \rightarrow W_2^1(\Omega), 1.2$$

where

$$a(u,v) = \int_\Omega \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx.$$

Let

$$V_h = \{v_h \in C(\overline{\Omega}) \mid v_h|_K \in P_1(K) \quad \forall K \in T_h\},$$

where T_h is a triangulation (decomposition) of $\overline{\Omega}$ into closed simplexes in the standard sense and $P_1(K)$ is the space of linear polynomials over K. Let $V_h^0 = V_h \cap \circ \rightarrow W_2^1(\Omega)$. A finite element approximation of (1.1) reads: Find $u_h \in V_h^0$ such that

$$a(u_h, v_h) = (f, v_h)_{0,\Omega}, \quad \forall v_h \in V_h^0 .1.3$$

Moreover, let $\pi_h : C(\overline{\Omega}) \rightarrow V_h$ be the usual linear interpolation operator such that

$$\pi_h v(Z) = v(Z), \quad \forall Z \in N_h,$$

where N_h is the set of nodes of T_h.

Recall that a family of triangulations $\mathcal{F} = \{T_h\}_{h\rightarrow 0}$ is said to be *regular* (*strongly regular*) if there exists a constant $\varkappa > 0$ such that for any $K \in T_h$ and any $T_h \in \mathcal{F}$ there exists a ball $\mathcal{B} \subset K$ with radius ρ_K such that $\varkappa h_K \leq \rho_K$ ($\varkappa h \leq \rho_K$), where $h_K = \text{diam } K$ and $h = \max_K h_K$.

We briefly recall the definition of the weighted averaged gradient introduced in details in [6]. For $Z \in N_h$ denote by $\ell_i = \ell_i(Z)$ that straight line which passes through Z and is parallel with the axis x_i.