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SERIES REPRESENTATION OF DAUBECHIES' WAVELETS*

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Abstract

This paper gives a kind of series representation of the scaling functions ϕ_N and the associated wavelets ψ_N constructed by Daubechies. Based on Poission summation formula, the functions $\phi_N(x+N-1),\ \phi_N(x+N),\cdots,\phi_N(x+2N-2)(0\le x\le 1)$ are linearly represented by $\phi_N(x),\phi_N(x+1),\cdots,\phi_N(x+2N-2)$ and some polynomials of order less than N, and $\mathbf{\Phi}_0(x):=(\phi_N(x),\phi_N(x+1),\cdots,\phi_N(x+N-2))^t$ is translated into a solution of a nonhomogeneous vector–valued functional equation

$$\mathbf{f}(x)=\mathbf{A}_d\mathbf{f}(2x-d)+\mathbf{P}_d(x),\ x\in[\frac{d}{2},\frac{d+1}{2}],\ d=0,1,$$

where $\mathbf{A}_0,\mathbf{A}_1$ are $(N-1)\times(N-1)$–dimensional matrices, the components of $\mathbf{P}_0(x),\mathbf{P}_1(x)$ are polynomials of order less than N. By iteration, $\mathbf{\Phi}_0(x)$ is eventualy represented as an $(N-1)$–dimensional vector series $\sum_{k=0}^{\infty}\mathbf{u}_k(x)$ with vector norm $\|\mathbf{u}_k(x)\|\le C\beta^k$, where $\beta=\beta_N<1$ and $\beta_N\searrow 0$ as $N\to\infty$.

1. Introduction.

In this paper we study the representation of Daubechies' wavelets. Daubechies[1] constructed a family of compactly supported regular scaling functions $\phi_N(x)$ and the associated regular wavelets $\psi_N(x)(N\ge 2)$:

$$\psi_N(x):=\sum_{n=-1}^{2N-2}(-1)^nC_N(n+1)\phi_N(2x+n),\qquad x\in\mathbf{R},\tag{1.1}$$

$$\phi_N(x):=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}\hat{\phi}_N(\xi)e^{-\mathrm{i}\xi x}d\xi,\qquad x\in\mathbf{R},\ \mathrm{i}=\sqrt{-1},$$

where $\hat{\phi}_N\in L^1(\mathbf{R})$ defined by

$$\hat{\phi}_N(\xi):=\frac{1}{\sqrt{2\pi}}\prod_{j=1}^{\infty}m_N(2^{-j}\xi),\ \hat{\phi}_N(0)=\frac{1}{\sqrt{2\pi}},$$

$$m_N(\xi):=\frac{1}{2}\sum_{n=0}^{2N-1}C_N(n)e^{\mathrm{i}n\xi}=\Big[\frac{1}{2}(1+e^{\mathrm{i}\xi})\Big]^N\sum_{k=0}^{N-1}q_N(k)e^{\mathrm{i}k\xi},\tag{1.2}$$

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the polynomial $\sum\limits_{k=0}^{N-1} q_{_N}(k)z^k$ satisfies

$$\left|\sum_{k=0}^{N-1} q_{_N}(k)e^{\mathrm{i}k\xi}\right|^2 = \sum_{k=0}^{N-1}\binom{k+N-1}{k}\sin^{2k}(\frac{\xi}{2}), \quad \xi\in\mathbf{R}, \tag{1.3}$$

with $\sum\limits_{k=0}^{N-1} q_{_N}(k)=1, q_{_N}(k)\in\mathbf{R}, k=0,1,\cdots,N-1$. It is known that[1] for each $N\geq 2$, $\operatorname{supp}\phi_{_N}=[0,2N-1]$, $\operatorname{supp}\psi_{_N}=[-(N-1),N]$ and the wavelet $\psi_{_N}$ generates by its dilations and translations an orthornormal basis $\{\sqrt{2^j}\psi_{_N}(2^jx-k)\}_{j,k\in\mathbf{Z}}$ of $L^2(\mathbf{R})$. The functions $\phi_{_N}$ and $\psi_{_N}$ have been proved to be very useful in numerical analysis[2,3]. On the aspect of representation, however, comparing to some nonorthogonal wavelets, the wavelets ψ_N and (any) other orthogonal regular wavelets seem to be hardly written in very explicit forms. This is not strange because for any wavelet ψ, its regularity, orthogonality (i.e. orthogonality of $\{\sqrt{2^j}\psi(2^jx-k)\}_{j,k\in\mathbf{Z}}$ in $L^2(\mathbf{R})$), symmetry, support compactness and representation (in the sense of computing) can not be satisfied simultaneously. So far there are two methods for approximating or representing the scaling functions ϕ_N, both of them are based on the two–scale difference equation[1,4,5]

$$\phi_N(x)=\sum_{n=0}^{2N-1} C_N(n)\phi_N(2x-n),\ x\in\mathbf{R}, \tag{1.4}$$

and homogeneous iterative approximation. One method is the iterative approximation scheme $f_n=Vf_{n-1}$, where V is a linear operator

$$Vf(x):=\sum_{k=0}^{2N-1} C_{_N}(k)f(2x-k)$$

acting on a function space. The $\phi_{_N}$ is therefore a fixed point of V, $V\phi_{_N}=\phi_{_N}$, computed by $\lim\limits_{n\to\infty}V^nf_0(x)=\phi_{_N}(x)$ with a suitable initial function f_0, e.g., interpolating spline. The convergence is uniform or pointwise depending on the choice of f_0[1,4]. Another method[5] is similar to that scheme but with vector (matrix) forms: Let $\mathbf{\Phi}(x)=(\phi_{_N}(x),\phi_{_N}(x+1),\cdots,\phi_{_N}(x+2N-2))^t, \mathbf{T}_0,\mathbf{T}_1\in\mathbf{R}^{(2N-1)\times(2N-1)}, (\mathbf{T}_d)_{ij}=C_{_N}(2i-j-1+d), d=0,1$ ($C_{_N}(n)=0\,\text{for}\,n<0\,\text{or}\,n>2N-1$). Then (1.4) is written $\mathbf{\Phi}(x)=\mathbf{T}_{d_1(x)}\mathbf{\Phi}(\tau(x))$, $x\in[0,1]$ since $\operatorname{supp}\phi_{_N}=[0,2N-1]$. Iteratively,

$$\mathbf{\Phi}(x)=\mathbf{T}_{d_1(x)}\mathbf{T}_{d_2(x)}\cdots\mathbf{T}_{d_n(x)}\mathbf{\Phi}(\tau^n(x)), x\in[0,1],$$

where the index $d_j(x)$ is the jth digit in the binary expansion for $x\in[0,1], \tau(x)$ is the shift operator: $\tau(x)=0.d_2(x)d_3(x)\cdots$, (see section 2). All the infinite products $\mathbf{T}_{d_1(x)}\mathbf{T}_{d_2(x)}\mathbf{T}_{d_3(x)}\cdots$ of the matrices $\mathbf{T}_0,\mathbf{T}_1$ are convergent in matrix norm and for a suitable initial function $\mathbf{v}_0(x)\in\mathbf{R}^{2N-1}$,

$$\mathbf{\Phi}(x)=\lim_{n\to\infty}\mathbf{T}_{d_1(x)}\mathbf{T}_{d_2(x)}\cdots\mathbf{T}_{d_n(x)}\mathbf{v}_0(\tau^n(x)),\ x\in[0,1]. \tag{1.5}$$

Both the schemes can achieve approximation degree as $O(2^{-\alpha n})(n\to\infty), \alpha>0$. In this paper we give a different method to represent (approximate) the scaling functions $\phi_{_N}$