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A LEGENDRE PSEUDOSPECTRAL METHOD FOR SOLVING NONLINEAR KLEIN-GORDON EQUATION*

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Abstract

A Legendre pseudospectral scheme is proposed for solving initial-boundary value problem of nonlinear Klein-Gordon equation. The numerical solution keeps the discrete conservation. Its stability and convergence are investigated. Numerical results are also presented, which show the high accuracy. The technique in the theoretical analysis provides a framework for Legendre pseudospectral approximation of nonlinear multi-dimensional problems.

1. Introduction

As we know, the Klein-Gordon equation is an important mathematical model in quantum mechanics. It is of the form

$$\begin{cases} \dfrac{\partial^2 U}{\partial t^2}(x,t) - \triangle U(x,t) + bU(x,t) + g(U(x,t)) = f(x,t), & x \in \Omega,\ 0 < t \le T, \\ U(x,t) = 0, & x \in \partial\Omega,\ 0 \le t \le T, \\ \dfrac{\partial U}{\partial t}(x,0) = U_1(x), & x \in \Omega, \\ U(x,0) = U_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega = (-1,1)^n$, $x = (x_1, x_2, \cdots, x_n)$, $\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$, $g(z) = |z|^\alpha z$, $p = \alpha + 2$ and b is a real number. Assume that $U_0(x) = U_1(x) = 0$ on $\partial\Omega$ and

$$\begin{cases} \alpha \ge 0, & \text{for } n \le 2, \\ 0 \le \alpha \le \dfrac{2}{n-2}, & \text{for } n \ge 3. \end{cases} \tag{1.2}$$

As in [1], it can be shown that if $U_0 \in H_0^1(\Omega) \cap L^p(\Omega)$, $U_1 \in L^2(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$, then (1.1) has unique solution $U \in C(0,T;H_0^1(\Omega) \cap L^p(\Omega))$. If U_0, U_1 and f are smoother, then U is smoother also. On the other hand, some finite difference schemes were proposed with strict proof of generalized stability and convergence. Their numerical solutions keep the discrete conservations. One of special cases ($\alpha = 2$) was considered also in [4]. But for all these finite difference approximations, the convergence rate is of order 2 in the space. To overcome it, some Fourier spectral and Fourier

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pseudospectral schemes were studied for periodic problems (see [5,6]). Their numerical solutions possess the convergence rate of "infinite order". Recently, Legendre spectral scheme was also studied for the initial-boundary value problem(see[7]). Its numerical results also show that it is more accurate than the corresponding finite difference scheme. However, because of the nonlinear term $g(U)$, it is very difficult to implement the spectral method strictly, unless α is a small integer. In this paper, we discuss the pseudospectral method for solving (1.1). In the next section, we construct a Legendre pseudospectral scheme which simulates the energy conservation law reasonably. In particular, it can be easily implemented for all α. We present the numerical results in section 3, which show the advantages of such approximation. Then we list some lemmas and prove the generalized stability and convergence in the last three sections. The technique in the theoretical analysis provides a framework for Legendre pseudospectral approximation of nonlinear multi-dimensional problems arising in quantum mechanics and other fields.

2. The Scheme

Let $L^q(\Omega) = \{v|v$ is Lebesgue measureable on Ω and $\|v\|_{L^q} < \infty\}$, where

$$\|v\|_{L^q(\Omega)} = \begin{cases} \Big(\int_\Omega |v|^q dx\Big)^{\frac{1}{q}}, & \text{if } 1 \le q < \infty, \\ ess \cdot \sup\limits_{x\in\Omega} |v(x)|, & \text{if } q = \infty. \end{cases}$$

For $q = 2$, we denote the inner product and the norm of $L^2(\Omega)$ by $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. Let Z be the set of all non-negative integers, and $\gamma_l \in Z$. Set $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$, $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ and $D^\gamma = \dfrac{\partial^{|\gamma|}}{\partial {x_1}^{\gamma_1} \partial {x_2}^{\gamma_2} \cdots \partial {x_n}^{\gamma_n}}$. For any non-negative integer m, define $H^m(\Omega) = \{v|D^\gamma v \in L^2(\Omega),\ 0 \le |\gamma| \le m\}$, with the semi-norm $|\cdot|_m$ and the norm $\|\cdot\|_m$ as follows

$$|v|_m = \Big(\sum_{|\gamma|=m} \|D^\gamma v\|^2\Big)^{1/2}, \quad ||v||_m = (||v||_{m-1}^2 + |v|_m^2)^{1/2}.$$

For non–negative real number s, we define $H^s(\Omega)$ by the interpolation between the spaces $H^{[s]}(\Omega)$ and $H^{[s+1]}(\Omega)$. Its norm and semi-norm are denoted by $||\cdot||_s$ and $|\cdot|_s$ respectively.

Let $j_l \in Z$, $j = (j_1, j_2, \cdots, j_n)$ and $|j| = \max\limits_{1\le l\le n} |j_l|$. Set $L_j(x) = \prod\limits_{l=1}^{n} L_{j_l}(x_l)$, $L_{j_l}(x_l)$ being the Legendre polynomial of degree j_l with respect to x_l. For Legendre spectral approximation in spatial directions, we define that for any positive integer N,

$$S_N = span\{L_j(x) \mid |j| \le N\}, \quad V_N = S_N \cap H_0^1(\Omega).$$

Let $P_N : L^2(\Omega) \longmapsto V_N$ be the L^2-orthogonal projection operator, i.e., for any $v \in L^2(\Omega)$, we have $(P_N v - v, \varphi) = 0$, $\forall \varphi \in V_N$.

In this paper, we consider the n-dimensional interpolation. Let $k_l \in Z$, $k = (k_1, k_2, \cdots, k_n)$, $|k| = \max\limits_{1\le l\le n} |k_l|$. Set $x^{(k)} = (x_1^{(k_1)}, x_2^{(k_2)}, \cdots, x_n^{(k_n)})$ and $\omega^{(k)} = \omega_1^{(k_1)}$