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THE OPTIMAL PRECONDITIONING IN THE DOMAIN DECOMPOSITION METHOD FOR WILSON ELEMENT*

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Abstract

This paper discusses the optimal preconditioning in the domain decomposition method for Wilson element. The process of the preconditioning is composed of the resolution of a small scale global problem based on a coarser grid and a number of independent local subproblems, which can be chosen arbitrarily. The condition number of the preconditioned system is estimated by some characteristic numbers related to global and local subproblems. With a proper selection, the optimal preconditioner can be obtained, while the condition number is independent of the scale of the problem and the number of subproblems.

1. The Construction of Preconditioner

Let Ω be a polygon domain in R^2, $f \in L^2(\Omega)$. Consider the homogeneous Dirichlet boundary value problem of Poisson equation,

$$\begin{cases} -\triangle u = f, & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

Assume that, for domain Ω, there are a coarser subdivision T_H with mesh size H and an another one T_h with mesh size h, which is obtained by refining T_H. The both subdivisions satisfy the quasi-uniformity and the inverse hypothesis.

For a given element T, $P_m(T)$ denotes the space of all polynomials with the degree not greater than m, $Q_m(T)$ denotes the space of all polynomials with the degree corresponding to x or y not greater than m.

Let V_H and V_h be some nonconforming finite element spaces corresponding to T_H and T_h respectively. For problem (1.1), the nodal parameters on the boundary $\partial\Omega$ are all zero. For finite element spaces V_h and V_H, the finite element equations for problem (1.1) are

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \tag{1.2}$$

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$$a_H(u^H, v^H) = (f, v^H), \quad \forall v^H \in V_H, \tag{1.3}$$

respectively. Where $(\cdot,\cdot)$ is $L^2(\Omega)$ inner product and

$$a_h(v, w) = \sum_{T \in \mathsf{T}_h} \int_T \Big(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \Big) dxdy,$$

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For $v \in V_h$, denote the vector of its nodal parameters by $C_h(v)$, and for $v \in V_H$, denote the vector of its nodal parameters by $C_H(v)$. Thus, equations (1.2) and (1.3) can be written as

$$A_h C_h(u_h) = F_h \tag{1.4}$$

$$A_H C_H(u^H) = F_H \tag{1.5}$$

where A_h, A_H are the stiffness matrices corresponding to problems (1.2) and (1.3) respectively, and F_h, F_H are the loading vectors.

Now consider how to solve (1.2). The Preconditioned Conjugate Gradient method (PCG) would be used. So the preconditioning matrix Q needs to be constructed.

Let $\{\omega_1, \omega_2, \cdots, \omega_M\}$ be a domain decomposition of Ω, i.e., $\overline{\Omega} = \cup_{k=1}^{M} \overline{\omega_k}$, and $\omega_m \cap \omega_n = \emptyset (m \neq n)$. For each ω_k, it is extended to Ω_k, such that the boundary of Ω_k is consists of the edges of T_h and

$$\text{dist}\,\{\partial\omega_k, \partial\Omega_k\} \geq L, \tag{1.6}$$

where L is a fixed positive constant. For each element $T \in \mathsf{T}_h$, the number of subdomains $\overline{\Omega}_k$ containing T does not exceed a fixed number.

Corresponding to T_h, a subdivision of Ω_k can be obtained, and the corresponding nonforming finite element space is denoted by $V_{h,k}$. The corresponding finite element equation is

$$a_k(u_k, v_k) = (f, v_k)_k, \quad \forall v_k \in V_{h,k}, \tag{1.7}$$

where $(\cdot,\cdot)_k$ is $L^2(\Omega_k)$ inner product and

$$a_k(u_k, v_k) = \sum_{T \in \mathsf{T}_h, T \subset \overline{\Omega}_k} \int_T \Big(\frac{\partial u_k}{\partial x}\frac{\partial v_k}{\partial x} + \frac{\partial u_k}{\partial y}\frac{\partial v_k}{\partial y} \Big) dxdy.$$

The stiffness matrix is denoted by A_k.

Let E_k be the zero extension operator from $V_{h,k}$ to V_h, i.e., $\forall v_k \in V_{h,k}, \forall T \in \mathsf{T}_h$

$$E_k v_k|_T = \begin{cases} v_k|_T, & T \subset \overline{\Omega}_k \\ 0, & \text{otherwise} \end{cases} \tag{1.8}$$