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CONVERGENCE OF A CONSERVATIVE DIFFERENCE SCHEME FOR THE ZAKHAROV EQUATIONS IN TWO DIMENSIONS*

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Abstract

A conservative difference scheme is presented for the initial-boundary-value problem of a generalized Zakharov equations. On the basis of a prior estimates in L_2 norm, the convergence of the difference solution is proved in order $O(h^2+r^2)$. In the proof, a new skill is used to deal with the term of difference quotient $(e^n_{j,k})t$. This is necessary, since there is no estimate of $E(x,y,t)$ in L_∞ norm.

1. Introduction

The Zakharov equations describe physical phenomena in Plasma[12]. The global existence of a weak solution for the Zakharov equations was considered by Sulem and Sulem in [11]. The existence and uniqueness of a smooth solution in one dimension are proved provided that smooth initial data are described. For small initial data, the existence of a weak solution for the Zakharov equations in two and three dimensions is obtained.

Numerical methods for the Zakharov equations in one dimension were considered in [1], [2], [4], [5] and [10]. A spectral method is used to compute solitary waves in [10]. In [4] and [5], Glassey considered an implicit difference scheme for the equations and proved its convergence in order $O(h+\tau)$. A new conservative difference scheme with a parameter $\theta, 0\le\theta\le\dfrac{1}{2}$ was presented in [2]. If $\theta=\dfrac{1}{2}$, the new scheme is identical to Glassey's scheme. For $\theta=0$ the new scheme is semi-explicit. In [1], we considered this semi-explicit scheme for generalized Zakharov equations and improved method of proof to get convergence in order$O(h^2+\tau^2)$. Numerical experiments demonstrate that the new scheme with $\theta=0$ is more accurate and efficient.

In this paper we consider the following periodic initial-value problem in two dimensions:

$$iE_t+E_{xx}+E_{yy}-NE=0, \quad \text{in } \Omega=(0,1)\times(0,1), \tag{1.1}$$

$$N_{tt}-N_{xx}-N_{yy}=(|E|^2)_{xx}, \quad \text{in } \Omega, \tag{1.2}$$

$$E|_{t=0}=E_0(x,y), \quad N|_{t=0}=N_0(x,y), \quad N_t|_{t=0}=N_1(x,y), \tag{1.3}$$

$$E(x+1,y,t)=E(x,y,t), \quad E(x,y+1,t)=E(x,y,t,)$$

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$$N(x+1,y,t) = N(x,y,t), \quad N(x,y+1,t) = E(x,y,t,), \tag{1.4}$$

where a complex unknown function E is the slowly varying envelope of highly oscillatory electric field and a real unknown function N denotes the fluctuation in the ion-density about its equilibrium value, $E_0(x,y), N_0(x,y)$ and $N_1(x,y)$ are periodic functions, $N_1(x,y)$ satisfies the compatibility condition:

$$\iint_\Omega N_1(x,y)dxdy = 0. \tag{1.5}$$

The periodic initial-value problem (1.1)–(1.5) possesses two conservative quantities:

$$\|E\|_{L^2}^2 = \text{const.} \tag{1.6}$$

and

$$\|E_x\|_{L^2}^2 + \|E_y\|_{L^2}^2 + \frac{1}{2}\|N\|_{L^2}^2 + \frac{1}{2}(\|u_x\|_{L^2}^2 + \|u_y\|_{L^2}^2) + \iint_\Omega N|E|^2 dxdy = \text{ Const.}, \tag{1.7}$$

where the potential function u is given by

$$u_{xx} + u_{yy} = N_t. \tag{1.8}$$

Assume that $E_0 \in H^1(\Omega), N_0 \in L_2(\Omega),\ N_1 \in H^{-1}(\Omega)$ and $\|E_0\|_{L_2} < \dfrac{1}{\sqrt{8}}$, then there exists a weak solution $E \in L^\infty(R^+,\ H^1(\Omega)),\ N \in L^\infty(R^+,\ L^2(\Omega))$ for the problem (1.1)–(1.5) (see [12]).

We propose an implicit conservative difference scheme for the problem (1.1)–(1.5) in this paper. We will prove the convergence of the difference solution in order $O(h^2+\tau^2)$. In the proof, a new skill is used to deal with the term $(e_{j,k}^n)_t$. this is necessary, since there is no estimate of $E(x,y,t)$ in L_∞ norm.

In section 2, we describe the difference scheme and its basic properties. Some prior estimates and proof of the convergence of the difference solution are given in Section 3.

2. Finite difference Scheme

In this section, the finite difference method for the problem (1.1)–(1.5) is considered. As usual, the following notations are used

$$\begin{aligned}
&h_x = \frac{1}{J}, \quad h_y = \frac{1}{K},\\
&x_j = jh_x, \quad y_k = kh_y, \quad t^n = n\tau,\\
&E(j,k,n) \equiv E(x_j, y_k, t^n), \quad N(j,k,n) \equiv N(x_j, y_k, t^n),\\
&E_{j,k} \sim E(j,k,n), \quad N_{j,k}^n \sim N(j,k,n),\\
&(W_{j,k}^n)_x = \frac{1}{h_x}(W_{j+1,k}^n - w_{j,k}^n), \quad (W_{j,k}^n)_{\bar{x}} = \frac{1}{h_x}(W_{j,k}^n - W_{j-1,k}^n),\\
&W_{j,k}^{n+\frac{1}{2}} = \frac{1}{2}(W_{j,k}^{n+1} + W_{j,k}^n), \quad \|W^n\|_2^2 = h_x h_y \sum_{j=1}^J \sum_{k=1}^K |W_{j,k}^n|^2,
\end{aligned}$$