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Unconstrained Methods for Generalized Complementarity Problems*1)

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Abstract

In this paper, the generalized complementarity problem is formulated as an unconstrained optimization problem. Our results generalize the results of [9]. The dimensionality of the unconstrained problem is the same as that of the original problem. If the mapping of generalized complementarity problem is differentiable, the objective function of the unconstrained problem is also differentiable. All the solutions of the original problem are global minimizers of the optimization problem. A generalized strict complementarity condition is given. Under certain assumptions, local properties of the correspondent unconstrained optimization problem are studied. Limited numerical tests are also reported.

1. Introduction

The complementarity problem, a special case of variational inequality problem, has many applications in different fields such as mathematical programming, game theory, economics. Generally, the standard complementarity problem has the following form:

$$y = F(x), \; x \geq 0, \; y \geq 0, \; \langle y, x \rangle = 0, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner products. When $F(x)$ is an affine function of x, it reduces to the *linear complementarity problem* which is denoted by LCP. Otherwise we call it *the nonlinear complementarity problem* or simply NCP. The complementarity problem has attracted many researchers since its appearance and many results have been given, a nice survey is given by [3]. The LCP problem, can be converted as a special linear programming or quadratic programming in the nonnegative orthant of R^n, thus many classical methods for linear programming are used to solve the LCP problem[10]. For the NCP problem, people often use so called NCP functions and formulate the NCP as a system of equations or unconstrained optimization problem, then classical methods for unconstrained optimization can be applied[4,7,8,11].

The generalized complementarity problem, denoted by $GCP(X, F)$, is to find a vector $x^* \in X$ such that:

$$F(x^*) \in X^*, \quad \text{and } \langle F(x^*), x^* \rangle = 0, \tag{1.2}$$

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where X is a convex cone in R^n, $X^* = -X^0$, and X^0 is the polar cone of X[12]

$$X^0 = \{y \in R^n : \langle y, x\rangle \leq 0, \ \forall\, x \in X\}. \tag{1.3}$$

It is obvious that when X is the nonnegative orthant of R^n, (1.2) reduces to the NCP. Although (1.2) was proposed and studied twenty-years ago [5], [6], little attention has paid to it. Traditionally it was considered as a variational inequality problem, which has the following standard form:

$$x^* \in X, \ \text{ and } \langle F(x^*), y - x^*\rangle \geq 0, \quad \forall\, y \in X, \tag{1.6}$$

which usually denoted by $VI(X, F)$. Then we use the same methods for VI problem to solve it. In doing so, through variational principle, a merit or gap function is applied, then we approach the solution of the GCP by minimizing the merit function.

Recently, some new interesting results for these problems are reported. In [2], through projection operators, the problem (1.4) is reconsidered as differentiable optimization problem. The objective function has some desirable global properties. In [9], for the NCP (1.1), the authors proposed unconstrained methods which mainly derived from an augmented Lagrangian formulation. Under certain conditions, the unconstrained problem has excellent local properties.

The main purpose of this paper is to generalize the results of [9] to the case where X is a convex cone. In the following section, we first describe some notations and concepts which will be used in the paper, similar to [9], we first consider a generalized augmented Lagrangian formulation. Then we show that this formulation equals to the difference of two functions defined in [2]. In section 3, some global properties of the optimization problem are discussed. A generalized strict complementarity condition is also considered. Under certain assumptions, we discuss the local properties of the optimization problem. Some numerical results are reported in the last section.

2. Preliminaries

First, we give some basic definitions [5], [6]:

Definition 2.1. *Let Ω be a nonempty subset of R^n; then*
(i): Ω is a cone if $x \in \Omega \Rightarrow \ \lambda x \in \Omega$ for all reals $\lambda \geq 0$;
(ii): Ω is a convex cone if $x \in \Omega, y \in \Omega \Rightarrow \lambda x + \mu y \in \Omega$ for all reals $\lambda \geq 0, \mu \geq 0$;
(iii): Ω is solid if it has nonempty interior relative to R^n.

In the rest of this paper, except for special description, we assume that the constraint set X is a closed solid convex cone, which means that problems (1.4) and (1.2) are equivalent. Similar to [5], [6], we define a partial ordering on R^n as follows: $x \overset{X}{\geq} y$ if and only if $x - y \in X$ and $x \overset{X}{>} y$ if and only if $x - y \in \operatorname{int}(X)$. Now we can reformulate (1.2) as the following constrained minimization problem:

$$\min_x \left\{ \langle F(x), x\rangle \middle| x \overset{X}{\geq} 0, F(x) \overset{X^*}{\geq} 0 \right\}. \tag{2.1}$$