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A NEW CLASS OF UNIFORMLY SECOND ORDER ACCURATE DIFFRENCE SCHEMES FOR 2D SCALAR CONSERVATION LAWS*

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Abstract

In this paper, concerned with the Cauchy problem for 2D nonlinear hyperbolic conservation laws,we construct a class of uniformly second order accurate finite difference schemes, which are based on the E-schemes. By applying the convergence theorem of Coquel-Le Floch [1], the family of approximate solutions defined by the scheme is proven to converge to the unique entropy weak L^∞-solution. Furthermore, some numerical experiments on the Cauchy problem for the advection equation and the Riemann problem for the 2D Burgers equation are given and the relatively satisfied result is obtained.

1. Convergence of A Class of Uniformly Second Order Accurate Difference Schemes

In this section, we consider the Cauchy problem for nonlinear hyperbolic scalar conservation laws with two space variables:

$$\partial_t u + \partial_x f(u) + \partial_y g(u) = 0, \quad u(t,x,y) \in R, t \in (0,T), (x,y) \in R^2, \tag{1.1}$$

$$u(0,x,y) = u_0(x,y), \quad (x,y) \in R^2, \tag{1.2}$$

where f and g: $R \to R$ are Lipschitz continuous functions and the initial data u_0 is a bounded function with compact support.

Let $\Delta t, \Delta x, \Delta y$ be the time, x-space and y-space increments of the discretization respectively. The mesh ratios, $\lambda_x = \Delta t/\Delta x$, $\lambda_y = \Delta t/\Delta y$, will be kept constants. $\Delta_+ u^n_{i+\frac{1}{2},j} = u^n_{i+1,j} - u^n_{i,j}$, $\Delta_+ u^n_{i,j+\frac{1}{2}} = u^n_{i,j+1} - u^n_{i,j}$.

In [2], the authors have discussed a class of high order accurate schemes constructed from E scheme by the flux limiters. The scheme is in the form ($n \in N$)

$$u^{n+1}_{i,j} = u^n_{i,j} - \lambda_x \Delta_+ f^n_{i+\frac{1}{2},j} - \lambda_y \Delta_+ g^n_{i,j+\frac{1}{2}}, \quad i,j \in Z, \tag{1.3}$$

$$f^n_{i+\frac{1}{2},j} = h\Big(u^n_{i+1,j} - \frac{1}{2} p^n_{i+\frac{1}{2},j},\ u^n_{i,j} + \frac{1}{2} q^n_{i+\frac{1}{2},j}\Big),$$

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$$g^n_{i,j+\frac{1}{2}} = l\Big(u^n_{i,j+1} - \frac{1}{2}r^n_{i,j+\frac{1}{2}},\ u^n_{i,j} + \frac{1}{2}s^n_{i,j+\frac{1}{2}}\Big), \quad i,j \in Z, \tag{1.4}$$

where

$$\begin{aligned}
p^n_{i+\frac{1}{2},j} &= \phi^1(t^n_{i+\frac{3}{2},j})\Delta_+u^n_{i+\frac{3}{2},j}\,\theta\Big(\frac{|\Delta_+u^n_{i+\frac{3}{2},j}|}{c_1h^{\alpha_1}}\Big)\\
q^n_{i+\frac{1}{2},j} &= \phi^1(w^n_{i+\frac{1}{2},j})\Delta_+u^n_{i-\frac{1}{2},j}\,\theta\Big(\frac{|\Delta_+u^n_{i-\frac{1}{2},j}|}{c_1h^{\alpha_1}}\Big)\\
r^n_{i,j+\frac{1}{2}} &= \phi^2(t^n_{i,j+\frac{3}{2}})\Delta_+u^n_{i,j+\frac{3}{2}}\,\theta\Big(\frac{|\Delta_+u^n_{i,j+\frac{3}{2}}|}{c_2h^{\alpha_2}}\Big)\\
s^n_{i,j+\frac{1}{2}} &= \phi^2(w^n_{i,j+\frac{1}{2}})\Delta_+u^n_{i,j-\frac{1}{2}}\,\theta\Big(\frac{|\Delta_+u^n_{i,j-\frac{1}{2}}|}{c_2h^{\alpha_2}}\Big), \quad n \in N, i,j \in Z, \qquad (1.5)\\
t^n_{i+\frac{1}{2},j} &= \frac{\Delta_+u^n_{i-\frac{1}{2},j}}{\Delta_+u^n_{i+\frac{1}{2},j}}, \quad t^n_{i,j+\frac{1}{2}} = \frac{\Delta_+u^n_{i,j-\frac{1}{2}}}{\Delta_+u^n_{i,j+\frac{1}{2}}}\\
w^n_{i+\frac{1}{2},j} &= \frac{1}{t^n_{i+\frac{1}{2},j}}, \quad w^n_{i,j+\frac{1}{2}} = \frac{1}{t^n_{i,j+\frac{1}{2}}}, \quad n \in N, i,j \in Z\\
\theta(r) &= \begin{cases} 1 & |r| \le 1 \\ bh & |r| > 1 \end{cases}, \quad b \ge 0\\
0 < \alpha^k &< 1, \quad c_k > 0, \quad \text{for } k = 1,2
\end{aligned}$$

$h(u,v)$, $l(u,v)$ are the numerical flux functions of any two three-point E-schemes. ϕ^1, ϕ^2 are flux limiters.

We list two results of the authors in [2] which will be needed in this paper.

Lemma 1.1. [2] *Suppose that the condition*

$$0 \le \phi^k(r) \le \mu, \quad \phi^k(0) = 0, \quad 0 \le \frac{\phi^k(r)}{r} \le 1, \quad \text{for } k = 1,2, \tag{1.6}$$

holds true and λ_x, λ_y *satisfy the condition*

$$\lambda_x \max_{u,v}\{|h_0|,|h_1|\} + \lambda_y \max_{u,v}\{|l_0|,|l_1|\} \le \frac{1}{2+\mu}, \tag{1.7}$$

where $h_0 = \partial h(u,v)/\partial v$, $h_1 = \partial h(u,v)/\partial u$, $l_0 = \partial l(u,v)/\partial v$, $l_1 = \partial l(u,v)/\partial u$. *Then the scheme* (1.3)–(1.5) *can be of the form* ($n \in N$)

$$u^{n+1}_{i,j} = u^n_{i,j} + C^n_{i+\frac{1}{2},j}\Delta_+u^n_{i+\frac{1}{2},j} - D^n_{i-\frac{1}{2},j}\Delta_+u^n_{i-\frac{1}{2},j} + E^n_{i,j+\frac{1}{2}}\Delta_+u^n_{i,j+\frac{1}{2}} - F^n_{i,j-\frac{1}{2}}\Delta_+u^n_{i,j-\frac{1}{2}},$$

where

$$C^n_{i+\frac{1}{2},j} \ge 0,\ D^n_{i+\frac{1}{2},j} \ge 0,\ E^n_{i,j+\frac{1}{2}} \ge 0,\ F^n_{i,j+\frac{1}{2}} \ge 0, \quad i,j \in Z, \tag{1.8}$$

$$C^n_{i+\frac{1}{2},j} + D^n_{i-\frac{1}{2},j} + E^n_{i,j+\frac{1}{2}} + F^n_{i,j-\frac{1}{2}} \le 1, \quad i,j \in Z. \tag{1.9}$$

Lemma 1.2. [2] *If the function* $\phi^k (k = 1,2)$ *satisfies* $\phi^k(x) = a^k_1 x + a^k_2$, *where* $a^k_1 \ge 0$, $a^k_2 \ge 0$, $a^k_1 + a^k_2 \equiv 1$, *for* $k = 1,2$, *then the scheme* (1.3)–(1.5) *is uniformly second order accurate in space.*