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MODELLING AND NUMERICAL SOLUTIONS OF A GAUGE PERIODIC TIME DEPENDENT GINZBURG-LANDAU MODEL FOR TYPE-II SUPERCONDUCTORS*1)

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Abstract

In this paper we seek the solutions of the time dependent Ginzburg-Landau model for type-II superconductors such that the associated physical observables are spatially periodic with respect to some lattice whose basic lattice cell is not necessarily rectangular. After appropriately fixing the gauge, the model can be formulated as a system of nonlinear parabolic partial differential equations with quasi-periodic boundary conditions. We first give some results concerning the existence, uniqueness and regularity of solutions and then we propose a semi-implicit finite element scheme solving the system of nonlinear partial differential equations and show the optimal error estimates both in the L^2 and energy norm. We also report on some numerical results at the end of the paper.

1. Introduction

Central to the theory of type-II superconductors is Abrikosov's characterization of the mixed state as a lattice-like arrangement of quantized flux lines, or vortices of superconducting electron pairs. The Abrikosov's vortex lattice, which has also been observed in experiments, is the solutions of the Ginzburg-Landau (GL) equations with a type of spatial periodicity. Recently there have been several authors studied the gauge periodic solutions of the *GL* superconductivity model from different point of views[1,10,11,17]. Roughly speaking, gauge periodic solutions are those solutions whose observables are spatially periodic with respect to some lattice (cf. §2). One of the key procedures in those studies is fixing the gauge. It is shown that after a preliminary gauge transformation, any gauge periodic solution can be assumed to have the form that the complex order parameter ψ satisfies some quasi-periodic boundary condition and the magnetic vector potential $\mathbf{A}$ is the sum of some periodic, divergence free function and $-\alpha G$ for some real constant α and $G(x) = (x_2, -x_1)^{\mathrm{T}}$.

The time-dependent Ginzburg-Landau (TDGL) model derived by Gor'kov and Èliashberg [15] from averaging the microscopic Bardeen-Cooper-Schrieffer theory offers a useful starting point in studying the dynamics of superconductivity. After appropriate nondimensionalization, the TDGL model can be formulated as in the following system of nonlinear partial differential equations (cf. e.g. [7], [3], [4]):

$$\eta \frac{\partial \psi}{\partial t} + i\eta\kappa\phi\psi + \Big(\frac{i}{\kappa}\nabla + \mathbf{A}\Big)^2 \psi + (|\psi|^2 - 1)\psi = 0, \tag{1.1}$$

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$$\frac{\partial \mathbf{A}}{\partial t} + \nabla \phi + \mathbf{curl}\,\mathrm{curl}\mathbf{A} + \Re\Big[\Big(\frac{i}{\kappa}\nabla\psi + \mathbf{A}\psi\Big)\bar{\psi}\Big] = 0, \tag{1.2}$$

where $\Re[\cdot]$ denotes the real part of the quantity in the brackets $[\cdot]$, and curl, **curl** denote the curl operators on $\mathbb{R}^2$ defined by

$$\mathrm{curl}\ \mathbf{A} = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}, \quad \mathbf{curl}\, v = \Big(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\Big)^{\mathrm{T}}.$$

Here ψ is a complex valued function and is usually referred to as the order parameter so that $|\psi|^2$ gives the relative density of the superconducting electron pairs; $\bar{\psi}$ is the complex conjugate of ψ; $\mathbf{A}$ is a real vector potential for the total magnetic field; ϕ is a real scalar function called electric potential; $\kappa > 0$ is the Ginzburg-Landau parameter which satisfies $\kappa > 1/\sqrt{2}$ for type-II superconductors; and $\eta > 0$ is a dimensionless constant.

The TDGL model (1.1)–(1.2) with Neumann boundary conditions has been studied in [5], [3], [7], [8], [12] with different gauge choices. The studies in [5] and [3] indicate that in contrast to the stationary case, where the Coloumb gauge div $\mathbf{A}$=0 is usually used, the Lorentz gauge div $\mathbf{A}+\phi = 0$ is more appropriate both in proving the regularity of solutions of the TDGL model and in designing numerically convergent algorithms solving the TDGL model. For more information about the subject of superconductivity, the reader may consult the two recent survey articles [2] and [9] and the references therein. We also refer to [10] for more discussions on the motivations of studying the periodic model for type-II superconductivity.

Our goal in this paper is to look for the solutions to (1.1)–(1.2) such that the associated physical observables (e.g. superconducting electron pairs, current, magnetic field, etc.) are spatially periodic, particularly for those solutions whose periodicity is supported on the hexagonal lattice. This problem was first considered in [13] on the rectangular lattice by using the Coloumb gauge. In that paper, the existence of weak solutions was obtained by using the method of lines and the problem of the asymptotic behavior for the time $t \to \infty$ was considered. In this paper, we first introduce the gauge periodic TDGL model, fix the gauge (Lorentz gauge) and then present some results concerning the existence, uniqueness and regularity of the solutions in §2. In §3 we propose a semi-implicit finite element scheme solving the gauge periodic TDGL model and in §4 we prove the optimal error estimates for the scheme both in the L^2 and energy norm. In §5 we report on a numerical example and in §6 we give some concluding remarks.

In the remainder of this section we introduce some of the notations to be used in the paper. Let $\mathcal{L}$ denote a planar lattice which consists of basis vectors $\mathbf{t}_1$ and $\mathbf{t}_2$. After rotation, we may always assume that the lattice $\mathcal{L}$ has a basis vector that is real. Thus we assume in the following that $\mathcal{L}$ is generated by $\mathbf{t}_1 = (r_1, 0)$ and $\mathbf{t}_2 = (r_2\cos\theta, r_2\sin\theta)$ with $r_1, r_2 > 0$ and $0 < \theta < \pi$. We denote Ω the open parallelogram generated by $\mathbf{t}_1$ and $\mathbf{t}_2$. In this paper we say that a function f is periodic if $f(x+\mathbf{t}_k) = f(x)$ for $k = 1, 2$ and a.e. $x \in \mathbb{R}^2$.

For any bounded open set $\mathcal{D} \subset \mathbb{R}^2$ and each integer $m \geq 0$ and real p with $1 \leq p \leq \infty$, we denote by $W^{m,p}(\mathcal{D})$ the standard Sobolev space of real functions having all their