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AN ACCURATE NUMERICAL SOLUTION OF A TWO DIMENSIONAL HEAT TRANSFER PROBLEM WITH A PARABOLIC BOUNDARY LAYER*

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Abstract

A singularly perturbed linear convection-diffusion problem for heat transfer in two dimensions with a parabolic boundary layer is solved numerically. The numerical method consists of a special piecewise uniform mesh condensing in a neighbourhood of the parabolic layer and a standard finite difference operator satisfying a discrete maximum principle. The numerical computations demonstrate numerically that the method is ε-uniform in the sense that the rate of convergence and error constant of the method are independent of the singular perturbation parameter ε. This means that no matter how small the singular perturbation parameter ε is, the numerical method produces solutions with guaranteed accuracy depending solely on the number of mesh points used.

Key words: Linear convection-diffusion, parabolic layer, piecewise uniform mesh, finite difference.

1. Introduction

Singularly perturbed differential equations are characterised by the presence of a small parameter ε multiplying the highest order derivatives. Such problems arise in many areas of applied mathematics. The solutions of singularly perturbed differential equations typically have steep gradients, in thin regions of the domain, whose magnitude depends inversely on some positive power of ε. Such regions are called either interior or boundary layers, depending on whether their location is the interior or the boundary of the domain. The location and width of these layers depend on the local asymptotic nature of the solution of the differential equation. Layers described by

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an ordinary, parabolic or elliptic differential equation are called respectively regular, parabolic or elliptic layers. Numerical methods for which the error bounds are independent of the singular perturbation parameter ε are called *ε-uniform* methods. In most previous work ε-uniform methods have been constructed and tested numerically only for singular perturbation problems with regular layers. In this paper, numerical results are presented for a singularly perturbed linear convection-diffusion problem with a parabolic layer. These results confirm numerically that numerical methods composed of a standard finite difference operator satisfying a maximum principle on a special piecewise-uniform mesh are ε-uniform. In fact it has been established theoretically in [9] that such numerical methods are ε-uniform for a wide class of singularly perturbed problems, including the problem considered here. Special piecewise-uniform meshes were first introduced and analyzed by Shishkin in [8]. The first computations using such methods were presented in [4].

2. Statement of the problem

Letting θ denote the temperature, $\vec{u} = (u_1, u_2)$ the velocity field of the fluid and $\varepsilon = \dfrac{1}{Pe}$ (where Pe is the Peclet number) the coefficient of diffusion, the transfer of heat in a two-dimensional region Ω is described by the following linear convection-diffusion equation (in dimensionless form)

$$\nabla \cdot (-\varepsilon \nabla \theta + \vec{u}\theta) = f \quad \text{in } \Omega \tag{2.1a}$$

where it is assumed that Ω is a bounded domain with Lipshitz continuous boundary Γ. Let Γ_D and Γ_N respectively denote the parts of Γ where Dirichlet and Neumann boundary conditions are specified, where $\Gamma = \Gamma_D \cup \Gamma_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. Let $\vec{n}$ denote the outward unit normal on Γ. The inflow and outflow boundaries Γ_i and Γ_o are defined respectively by

$$\Gamma_o = \{(x, y) \in \Gamma : (\vec{u} \cdot \vec{n})(x, y) > 0\}, \qquad \Gamma_i = \{(x, y) \in \Gamma : (\vec{u} \cdot \vec{n})(x, y) < 0\}$$

It is assumed that $\Gamma_D \supset \Gamma_i$, that the diffusion coefficient ε is positive and that $\nabla \cdot \vec{u} = 0$. The latter condition means that the velocity of the fluid $\vec{u}$ corresponds to an incompressible flow. When $\varepsilon \ll 1$, the differential equation (2.1a) is singularly perturbed and the flow is said to be convection dominated.

The solution of this linear problem with general boundary conditions can be decomposed into the sum of a smooth and a singular part for each kind of singularity. In this paper we choose boundary conditions so that there is just one kind of singularity, namely a parabolic boundary layer. The purpose of this paper is to obtain ε-uniformly accurate solutions for this special problem. Because of the linearity, it is clear that the same numerical method will solve any general linear problem having this type of singularity.