Journal of Computational Mathematics, Vol.16, No.1, 1998, 51–62.

ASYMPTOTIC ERROR EXPANSION AND DEFECT CORRECTION FOR SOBOLEV AND VISCOELASTICITY TYPE EQUATIONS

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Abstract

In this paper we study the higher accuracy methods − the extrapolation and defect correction for the semidiscrete Galerkin approximations to the solutions of Sobolev and viscoelasticity type equations. The global extrapolation and the correction approximations of third order, rather than the pointwise extrapolation results are presented.

Key words: Asymptotic error, semidiscrete Galerkin approximation, global extrapolation, higher accuracy.

1. Introduction

Let Ω be a rectangular domain. We are concerned with the Richardson extrapolation and defect correction of the finite element approximations to the solutions of the following simple Sobolev type equation

$$\begin{cases} -\triangle u_t - \triangle u = f & \text{in} \quad \Omega \times (0,T], \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,y,0) = v(x,y) & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

and viscoelasticity type equation

$$\begin{cases} u_{tt} - \triangle u_t - \triangle u = f & \text{in} \quad \Omega \times (0,T], \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T], \\ u(x,y,0) = v(x,y), \ u_t(x,y,0) = w(x,y) & \text{in} \quad \Omega. \end{cases} \tag{1.2}$$

The extrapolation technique used for the finite element approximations to the solutions of the elliptic differential equations has well been investigated in [1-6, 13-17, 21, 23]. And this method has also been considered for boundary element approximations to boundary integral equations (e.g. see [25-26]). Some further investigations of the extrapolation for Galerkin method for parabolic equations have been carried out in [7-9]. Not long ago, multi-parameter parallel algorithms have been introduced into the extrapolation in order to accelerate the computational speeds [27].

* Received June 19, 1995.

Defect correction schemes of the finite element approximations for the elliptic problems have been investigated by some authors, too. For example, see [2,19,20,24].

The problems (1.1) and (1.2) can arise from many physical processes. Many authors have studied both the finite difference and the finite element methods for these problems. Especially, the error estimates of FEM for the problems (1.1) and (1.2) have been deliberated in [11] via the Ritz-Volterra projection. Here, we will use a new analysis in [12], i.e. an analysis for the "short side" in the FE-right triangle together with the sharp integral estimates of the "hypotenuse" to present an immediate analysis for extrapolation and correction for (1.1) and (1.2). Moreover, we obtain global extrapolation results by means of an interpolation postprocessing technique proposed in [18] (or [22]), instead of the pointwise extrapolation results.

2. Sobolev Type Equations

Above all, we discuss the problem (1.1). Throughout the paper, we assume that T^h is a rectangular partition over Ω with mesh size h. The weak form of (1.1) consists in finding $u \in H_0^1(\Omega)$ (the Sobolev space) such that

$$\begin{cases} (\nabla u_t, \nabla\varphi) + (\nabla u, \nabla\varphi) = (f, \varphi) \quad \forall\varphi \in H_0^1, \\ u(0) = v. \end{cases} \tag{2.1}$$

Let $S_0^h \subset H_0^1$ consist of piecewise bilinear functions. Thus, a continuous Galerkin approximation $u^h : [0,T] \to S_0^h$ is defined such that

$$\begin{cases} (\bigtriangledown u_t^h, \bigtriangledown\varphi) + (\bigtriangledown u^h, \bigtriangledown\varphi) = (f, \varphi) \quad \forall\varphi \in S_0^h, \\ u^h(0) = i_h v, \end{cases} \tag{2.2}$$

where $i_h v \in S_0^h$ is the bilinear interpolation function of v. And thus, from (2.1) and (2.2) we get the error equation

$$(\nabla(u_t^h - u_t), \nabla\varphi) + (\nabla(u^h - u), \nabla\varphi) = 0 \quad \forall\varphi \in S_0^h. \tag{2.3}$$

Set $E(x) = \frac{1}{2}[(x - x_\tau)^2 - h_\tau^2]$, $F(y) = \frac{1}{2}[(y - y_\tau)^2 - k_\tau^2]$, associated with any element $\tau = [x_\tau - h_\tau, x_\tau + h_\tau] \times [y_\tau - k_\tau,\, y_\tau + k_\tau]$ of T^h. Then, there holds by [22] for $\varphi \in S_0^h$,

$$\int_\Omega \nabla(u - i_h u)\nabla\varphi = \int_\Omega \Big\{\Big[F\varphi_x - \frac{1}{3}(F^2)_y\varphi_{xy}\Big]u_{xyy} + \Big[E\varphi_y - \frac{1}{3}(E^2)_x\varphi_{xy}\Big]u_{yxx}\Big\}. \tag{2.4}$$

What is more, for $\varphi \in S_0^h$

$$\int_\Omega \nabla(u - i_h u)\nabla\varphi = -\int_\Omega \Big[F\varphi - \frac{1}{3}(F^2)_y\varphi_y + E\varphi - \frac{1}{3}(E^2)_x\varphi_x\Big]u_{xxyy}. \tag{2.5}$$

Lemma 2.1. *For* $\varphi \in S_0^h$,

$$(\nabla(u - i_h u), \nabla\varphi) = \frac{h^2}{3}\sum_\tau \frac{h_\tau^2 + k_\tau^2}{h^2}\int_\tau \varphi u_{xxyy} + O(h^3)\|u\|_5\|\varphi\|_0.$$