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A FAMILY OF DIFFERENCE SCHEMES WITH FOUR NEAR-CONSERVED QUANTITIES FOR THE KdV EQUATION*[1)]

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Abstract

We construct and analyze a family of semi-discretized difference schemes with two parameters for the Korteweg-de Vries (KdV) equation. The scheme possesses the first four near-conserved quantities for periodic boundary conditions. The existence and the convergence of its global solution in Sobolev space $\mathbf{L}_\infty(0,T;\mathbf{H}^3)$ are proved and the scheme is also stable about initial values. Furthermore, the scheme conserves exactly the first two conserved quantities in the special case.

Key words: Convergence, difference scheme, KdV equation, conserved quantity

1. Introduction

In this paper, we are concerned with the semi-discretized difference methods which are capable of approximating to the KdV equation to a considerable extent. Consider the periodic initial-boundary problem:

$$u_t + uu_x + u_{xxx} = 0, \qquad -\infty < x < +\infty, \quad t > 0 \tag{1.1}$$

$$u(x+1,t) = u(x,t), \qquad -\infty < x < +\infty, \quad t > 0 \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad -\infty < x < +\infty \tag{1.3}$$

where $u_0(x)$ is a given 1-periodic function and belongs to $\mathbf{H}^3$. Let J be a positive integer, put the spatial mesh length $h = 1/J$. Discrete periodic function $V_h = \{V_j | j = 0, \pm1, \pm2, \cdots\}$ takes the values on the net points $x_j = jh$. Denote Δ_o, Δ_+ and Δ_-, respectively, of the centered, the forward and the backward difference quotient operators, i.e.,

$$\Delta_o V_j = \frac{V_{j+1} - V_{j-1}}{2h}, \quad \Delta_+ V_j = \frac{V_{j+1} - V_j}{h}, \quad \Delta_- V_j = \frac{V_j - V_{j-1}}{h} \tag{1.4}$$

and E is a mean operator as follows

$$EV_j = \frac{1}{2}(V_{j+1} + V_{j-1}). \tag{1.5}$$

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As similar as [13], for real $1 \le p \le \infty$, denote by $\mathbf{W}_p = \mathbf{W}_p(0,1)$ the usual discretized real Sobolev spaces on (0, 1) and by $\|\cdot\|_p$ the associated norm:

$$\|V_h\|_p = \Big(\sum_{j=1}^{J} |V_j|^p h\Big)^{1/p}, \quad 1 \le p \le \infty. \tag{1.6}$$

For integer $s \ge 0$, let $\mathbf{H}^s = \mathbf{W}_2^s$ and the inner product on $\mathbf{W}_2(0,1)$ is denoted by $(\cdot,\cdot)$.

The KdV equation (1.1) can be shown to have an infinite hierarchy of conservation quantities and the first four of them can be written as follows[7]:

$$F_0(u) = \int_0^1 3u\mathrm{d}x \tag{1.7a}$$
$$F_1(u) = \int_0^1 \frac{1}{2}u^2\mathrm{d}x \tag{1.7b}$$
$$F_2(u) = \int_0^1 \Big(\frac{1}{6}u^3 - \frac{1}{2}u_x^2\Big)\mathrm{d}x \tag{1.7c}$$
$$F_3(u) = \int_0^1 \Big(\frac{5}{70}u^4 - \frac{5}{6}uu_x^2 + \frac{1}{2}u_{xx}^2)\mathrm{d}x. \tag{1.7d}$$

Unfortunately, it is difficult for discretizations of (1.1) to preserve more than two exact conserved quantities. Although the numerical studies of the KdV equation have been largely developed since Zabusky and Kruskal used the second order accuracy Leap-Frog scheme to solve this evolution equation, there were seldom works discussing about multiple conservation laws of difference approximation of (1.1) or estimates of difference solutions under norm $\|\cdot\|_\infty$ and their higher order difference quotients. Recently, the various computational instabilities occurring in difference approximating to the KdV equation were observed by several scholars[1,10,11]. One was conscious that high order discrete conserved quantities are very significant for restraining numerical instabilities.

In authors' previous papers [3], [4] and [5], several semi-discrete difference schemes were studied. They have three or four near-conserved quantities. Recently, we presented a method in [9] to construct schemes with multiple near-conserved quantities. The method draws construction of infinite conservation laws in continuous situation, which can be refered in Lax [7], and utilizes discretizations of the gradients of the invariant functionals. By the way, we obtained a family of schemes with a real parameter β:

$$\begin{aligned} V_{jt} &+ \frac{1}{2}\Delta_o V_j^2 + \Delta_o\Delta_+\Delta_- V_j + \frac{1-\beta}{12}h^2\Delta_o V_j\Delta_+\Delta_- V_j \\ &+ \frac{\beta}{6}h^2 V_j\Delta_o\Delta_+\Delta_- V_j + \frac{\beta-2}{36}h^4\Delta_+\Delta_- V_j\Delta_o\Delta_+\Delta_- V_j = 0 \end{aligned} \tag{1.8}$$

for J odd, the condition required by the inverse operation of difference operators. The schemes presented in [3] and [5] are the special cases of (1.8) for $\beta = 1$ and $\beta = 0$ respectively.

In this paper, we prove that the scheme (1.8) possesses the first four near-conserved quantities for J any positive integer. We gain the estimates of the difference solution and its difference quotients up to order 3 using the theory of discrete functional analysis