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A PARALLEL ITERATIVE DOMAIN DECOMPOSITION ALGORITHM FOR ELLIPTIC PROBLEMS

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Abstract

An iterative nonoverlapping domain decomposition procedure is proposed and analyzed for linear elliptic problems. At the interface of two subdomains, one subdomain problem requires that Dirichlet data be passed to it from the previous iteration level, while the other subdomain problem requires that Neumann data be passed to it. This procedure is suitable for parallel processing. A convergence analysis is established. Standard and mixed finite element methods are employed to give discrete versions of this domain decomposition algorithm. Numerical experiments are conducted to show the effectiveness of the method.

Key words: Domain decomposition methods, finite element methods, parallel computing

1. Introduction

Nonoverlapping domain decomposition methods have received a lot of attention during the past few years, since they have advantages of dealing with transmission problems and allow efficient parallelism. For a recent development of these methods, we refer to the papers by Funaro, Quarteroni and Zanolli[7], Marini and Quarteroni[11,12], Lions[10], Després[4], Douglas, Paes Leme, Roberts and Wang[5], and the author[13,14].

In this paper, we propose an iterative nonoverlapping domain decomposition procedure for second order partial differential equations. At the interface of two subdomains, one subdomain problem requires that Dirichlet data be passed to it from the previous iteration level, while the other subdomain problem requires that Neumann data be passed to it. Thus, this procedure can be efficiently implemented on computers with parallel architecture, as an improvement of the method in [7], [11], [12]. Both the method and the convergence analysis in this paper are closely related to and based on the techniques given in [7], [11], [12]. However, we will introduce a Galerkin approximation with Lagrange multipliers and a hybridized mixed finite element method, which were not dealt with in [7], [11], [12]. We will also prove that the error reduction factors per iteration in Galerkin approximations and hybridized mixed finite element approximations are independent of the grid size.

In §2, the domain decomposition method is described for general elliptic problems. In §3, a convergence analysis is carried out for general linear elliptic problems in multidimensions. In §4, a finite element approximation is employed. In §5, a finite element approximation with Lagrange multipliers is considered. Then, in §6, a hybridized mixed

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finite element method is applied. Finally in §7, numerical experiments are provided to check the correctness of the theory.

2. Domain Decomposition Method

Let Ω be a smooth bounded domain or a convex polygon in $\mathcal{R}^2$ with boundary $\partial\Omega$. Consider the following boundary value problem: find $u \in H^1(\Omega)$ such that

$$Lu = f \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega, \tag{1}$$

where $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$ are given, and the operator L is defined by

$$Lu = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + a_0(x)u. \tag{2}$$

The coefficients $\{a_{ij}\}$ are assumed to be symmetric, uniformly positive definite, bounded, and piecewise smooth in Ω, and $a_0 \geq 0$.

For simplicity, we partition the domain Ω into two nonoverlapping subdomains Ω_1 and Ω_2 such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2$, $\Omega_1 \cap \Omega_2 = 0$ We denote the interface by $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. The following argument makes it possible to include more than two computational subdomains: If either Ω_1 or Ω_2 is not a connected set, then decompose them into connected components:

$$\Omega_1 = \bigcup_{j=1}^{N_1} \Omega_{1j}, \quad \Omega_2 = \bigcup_{j=1}^{N_2} \Omega_{2j}, \tag{3}$$

where N_1 and N_2 are some positive integers. However, we must confine ourselves to the case in which no interior vertices are allowed. That is, only strip-type domain decompositions are considered here. Domain decompositions with cross points will be treated later [6].

We now define the following domain decomposition method: Choose $u_k^0 \in H^1(\Omega_k)$ with $u_k^0|_{\partial\Omega_k \cap \partial\Omega} = g$, $k = 1, 2$. For $n = 0, 1, 2, \cdots$, we construct the sequence $u_k^{n+1} \in H^1(\Omega_k)$ with $u_k^{n+1}|_{\partial\Omega \cap \partial\Omega_k} = g$ satisfying

$$Lu_1^{n+1} = f \text{ in } \Omega_1, \qquad \frac{\partial u_1^{n+1}}{\partial \nu_A^1} = \theta\frac{\partial u_1^n}{\partial \nu_A^1} + (1-\theta)\frac{\partial u_2^n}{\partial \nu_A^1} \text{ on } \Gamma, \tag{4}$$

$$Lu_2^{n+1} = f \text{ in } \Omega_2, \qquad u_2^{n+1} = \delta u_1^n + (1-\delta)u_2^n \text{ on } \Gamma, \tag{5}$$

where $\frac{\partial u_k^n}{\partial \nu_A^k} = \sum_{i,j=1}^{2} a_{ij}\frac{\partial u_k^n}{\partial x_j}\nu_i^k$, $\nu^k = \{\nu_1^k, \nu_2^k\}$ is the outward unit normal vector to $\partial\Omega_k$, and $\theta, \delta \in (0, 1)$ are relaxation parameters that will be determined to ensure and to accelerate the convergence of the iterative procedure.

The differences between this method and the one in [7], [11], [12] lie in that the former method gives parallelizable subdomain problems at each iteration level, while the latter method leads to sequential subdomain problems, and that the former utilizes the Neumann boundary values on an interface from two neighboring subdomains, while