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VARIATIONS ON A THEME BY EULER[*1)]

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Abstract

The oldest and simplest difference scheme is the explicit Euler method. Usually, it is not symplectic for general Hamiltonian systems. It is interesting to ask: Under what conditions of Hamiltonians, the explicit Euler method becomes symplectic? In this paper, we give the class of Hamiltonians for which systems the explicit Euler method is symplectic. In fact, in these cases, the explicit Euler method is really the phase flow of the systems, therefore symplectic. Most of important Hamiltonian systems can be decomposed as the summation of these simple systems. Then composition of the Euler method acting on these systems yields a symplectic method, also explicit. These systems are called symplectically separable. Classical separable Hamiltonian systems are symplectically separable. Especially, we prove that any polynomial Hamiltonian is symplectically separable.

Key words: Hamiltonian systems, symplectic difference schemes, explicit Euler method, nilpotent, symplectically separable

1. Introduction

A Hamiltonian system of differential equations on $\mathbf{R}^{2n}$ is given by

$$\dot{p} = -H_q(p,q), \quad \dot{q} = H_p(p,q), \tag{1}$$

where $p = (p_1, \cdots, p_n)$, $q = (q_1, \cdots, q_n) \in \mathbf{R}^n$ are the generalized coordinates and momenta respectively and $H(p,q)$ is the energy of the system. The system (1) can be rewritten as the compact form

$$\dot{z} = JH_z(z), \quad J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}, \tag{2}$$

where $z = (z_1, \cdots, z_n, z_{n+1}, \cdots, z_{2n}) = (p,q) \in \mathbf{R}^{2n}$, $H(z) = H(p,q)$. The phase flow, denoted by e_H^t, of the Hamiltonian system is symplectic, i.e., it preserves the differential 2-form on $\mathbf{R}^{2n}$ ${e_H^t}^*\omega = \omega$, where $\omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$, or $(e_H^t)_z^T(z)J(e_H^t)_z(z) = J$, $\forall z \in \mathbf{R}^{2n}$.

For the Hamiltonian system (2), a single step numerical method can be characterized by a map g_H^τ, τ is the time step size, $z^{n+1} = g_H^\tau z^n$, or $\hat{z} = g_H^\tau z$. If g_H^τ is symplectic, i.e., $(g_H^\tau)_z^T(z)J(g_H^\tau)_z(z) = J$, $\forall z \in \mathbf{R}^{2n}$, then, the method g_H^τ is called symplectic.

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The oldest and simplest difference scheme for Hamiltonian system (2) is the explicit Euler method

$$\hat{z} = E_H^\tau z := z + \tau J H_z(z), \quad E_H^\tau = 1 + \tau J H_z. \tag{3}$$

Usually, it is not symplectic for general Hamiltonian systems. But, it is symplectic for a kind of specific Hamiltonian systems, i.e., systems with nilpotent of degree 2 (see Section 2). In fact, it is the exact phase flow for these systems, therefore is symplectic. Many important Hamiltonian systems can be decomposed as the summation of Hamiltonian systems with nilpotent of degree 2, which are called symplectically separable. Then explicit symplectic schemes can be derived by composition of explicit Euler methods acting on these systems (exact phase flows). This kind of Hamiltonians is not too rare but can cover most important cases. Usual Hamiltonian systems in classical mechanics are symplectically separable. Especially, classical separable Hamiltonian systems are symplectically separable. At last, we proved that any polynomial Hamiltonian is symplectically separable.

This paper is devoted to the construction of explicit symplectic algorithms for symplectically separable Hamiltonian systems. In section 2, we give the definition of symplectically separable Hamiltonian systems. We list possible symplectically separable Hamiltonian systems in section 3 and give the construction of explicit symplectic algorithms for these systems in section 4. In section 5 we prove that all polynomials in $\mathbf{R}^{2n}$ are symplectically separable, therefore there exist explicit symplectic algorithms for these systems in principle. More detail materials about Hamiltonian systems, symplectic geometry and symplectic algorithms can be referred to [1–23].

2. Systems with Nilpotent of Degree 2

Definition 1. A Hamiltonian H is nilpotent of degree 2 if H satisfies

$$J H_{zz}(z) J H_z(z) = 0, \quad \forall z \in \mathbf{R}^{2n}. \tag{4}$$

Evidently, $H(p,q) = \phi(p)$ or $H(p,q) = \psi(q)$, which presents inertial flow and standing flow, are nilpotent of degree 2 since for $H(p,q) = \phi(p)$,

$$H_{zz}(z) J H_z(z) = \begin{bmatrix} \phi_{pp} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} \phi_p \\ 0 \end{bmatrix} = \begin{bmatrix} \phi_{pp} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \phi_p \end{bmatrix} = 0$$

and for $H(p,q) = \psi(q)$,

$$H_{zz}(z) J H_z(z) = \begin{bmatrix} 0 & 0 \\ 0 & \psi_{qq} \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \psi_q \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \psi_{qq} \end{bmatrix} \begin{bmatrix} -\psi_q \\ 0 \end{bmatrix} = 0.$$

Theorem 1. *If H is nilpotent of degree 2 then the explicit Euler method E_H^τ is the exact phase flow of the Hamiltonian, therefore symplectic.*

Proof. Let $z = z(0)$. From the condition (4) it follows that

$$\ddot{z}(t) = \frac{d}{dt} J H_z(z(t)) = (J H_z(z(t)))_z \dot{z}(t) = J H_{zz}(z(t)) J H_z(z(t)) = 0.$$