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A MULTI-PARAMETER SPLITTING EXTRAPOLATION AND A PARALLEL ALGORITHM FOR ELLIPTIC EIGENVALUE PROBLEM*

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Abstract

The finite element solutions of elliptic eigenvalue equations are shown to have a multi-parameter asymptotic error expansion. Based on this expansion and a splitting extrapolation technique, a parallel algorithm for solving multi-dimensional equations with high order accuracy is developed.

Key words: Finite element, multi-parameter error expansion, parallel algorithm, splitting extrapolation.

1. Introduction

The extrapolation method has become an important technique to obtain more accurate numerical solutions since it was first established by Richardson in 1926. The applications of extrapolation method in the finit difference can be found in [14]. In 1983, Q.Lin, T.Lü and S.Shen[8] introduced this technique into the finite element method, the development in this direction can be found in [5, 11, 12, 16]. However, this technique has a limitation when dealing with high dimentional problems, since the increasing of the dimension will cause an enormous amount of grid points which requires great computer power in case of the successive refinement. Recently, Zhou et al.[19,20] introduce a so called multi-parameter splitting extrapolation method. In this new method, the domain is divided into several subdomains based on the geometry of the domain, each of which is covered by different meshes so that the number of independent mesh parameters, say p, is as large as possible, and a higher order accuracy approximation is produced by $(p+1)$-processors in parallel. In general, p can be greater than the dimension of the problem. As a result, if the size of the original discrete problem is large, then the size of problems to be dealt with in each processor can be reduced significantly. In this paper, we adopt this method to the elliptic eigenvalue problem, a parallel algorithm for higher order approximations is also proposed.

2. Multi-Parameter Asymptotic Expansion for Eigenvalue

In this section, we only investigate simple eigenvalue problems for elliptic equations, so that we can concentrate on the main idea behind the construction without involving much effort in less important things, let us consider the Dirichlet problem

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$$\begin{cases} -\triangle u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega = (0,1)^n \subset R^n (n \geq 2)$.

Its weak form reads as follow: find $(\lambda, u) \in R \times (H_0^1(\Omega)\backslash\{0\})$ such that

$$a(u,v) = \lambda(u,v), \qquad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where $a(u,v) = \int_\Omega \nabla u \nabla v$, and $(f,v) = \int_\Omega fv$, $\int_\Omega \cdot = \int_\Omega \cdot dx_1 \cdots dx_n$.

Let Ω be divided into m rectangular subdomains $T = \{\Omega_j : j = 1, 2, \cdots, m\}$ so that the edges of each subdomain are parallel to the coordinate axe respectively and T is quasi-uniform. On the subdomain Ω_j, a rectangular mesh refinement with mesh parameters $\{h_{j,1}, \cdots, h_{j,n}\}$ is imposed, where $2h_{j,i}$ is the mesh size in the i^{th} coordinate direction. Assume that the union of all meshes form a quasi-uniform $n-$rectangular partition T^h of Ω with size h, then T^h is determined by some mesh parameters, say $h_1, \cdots, h_p$, with $h = \max\{h_i : i = 1, \cdots, p\}$ and $n \leq p \leq n + m - 1$. To minimize the sizes of the discrete subproblems, p may be chosen such that $p > n$.

Let $S^h(\Omega) = \{v \in C(\Omega)$: $v|_e$ is n-linear, $\forall e \in T^h\}$, $S_0^h(\Omega) = S^h(\Omega) \cap H_0^1(\Omega)$, and $i_h : C(\Omega) \longrightarrow S^h(\Omega)$ be the usual $n-$linear interpolation operator.

The finite element approximation of eigenvalue problem is determined by finding $(\lambda_h, u_h) \in R \times (S_0^h(\Omega)\backslash\{0\})$ satisfying

$$a(u_h, \varphi) = \lambda_h(u_h, \varphi), \quad \forall \varphi \in S_0^h(\Omega). \tag{2.3}$$

For continuous eigenvalue λ, there holds an orthonormal eigenfunction u and discrete solutions $(\lambda_h, u_h) \in R \times S_0^h(\Omega)$ such that

$$|\lambda - \lambda_h| + \|u - u_h\|_{0,2} \leq ch^2 \|u\|_{2,2}, \tag{2.4}$$

where $\|\cdot\|_{0,2}$ denotes the usual Soblev space, we also denote it by $\|\cdot\|$ in the following. For simplicity, assume that $T^h|_{\Omega_i}$ is uniform and u is smooth enough. We denote $R_h u$ to be the Ritz projection of u which is determined by the equation

$$\int_\Omega \bigtriangledown(u - R_h u) \bigtriangledown v = 0, \quad \forall v \in S_0^h(\Omega).$$

For $e \in T^h$, denote the center of e by $x_e = (x_{e,1}, \cdots, x_{e,n})$ and $e = \prod_{j=1}^n [x_{e,j} - h_{e,j}, x_{e,j} + h_{e,j}]$. For $1 \leq j \leq n$, define

$$F_{e,j}(x_j) = \frac{1}{2}((x_j - x_{e,j})^2 - h_{e,j}^2).$$

From the definition of $F_{e,j}(x_j)$, we easily get the following useful identity

$$F_{e,j} = \frac{1}{6}(F_{e,j}^2)'' - \frac{1}{3}h_{e,j}^2. \tag{2.5}$$

We recall that there holds the following multi-parameter expantion (cf.[19,20]).