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A QUASI-NEWTON METHOD IN INFINITE-DIMENSIONAL SPACES AND ITS APPLICATION FOR SOLVING A PARABOLIC INVERSE PROBLEM*[1)]

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Abstract

A Quasi-Newton method in Infinite-dimensional Spaces (QNIS) for solving operator equations is presented and the convergence of a sequence generated by QNIS is also proved in the paper. Next, we suggest a finite-dimensional implementation of QNIS and prove that the sequence defined by the finite-dimensional algorithm converges to the root of the original operator equation providing that the later exists and that the Fréchet derivative of the governing operator is invertible. Finally, we apply QNIS to an inverse problem for a parabolic differential equation to illustrate the efficiency of the finite-dimensional algorithm.

Key words: Quasi-Newton method, parabolic differential equation, inverse problems in partial differential equations, linear and Q-superlinear rates of convergence

1. Introduction

Quasi-Newton methods play an important role in numerically solving non-linear systems of equations on the Euclidean spaces. But it seems that the quasi-Newton methods have not been applied directly to solving inverse problems in partial differential equations (PDE) up to now if we exclude those methods, by which inverse problems in PDEs are formulated as optimization problems with equality constraints.

We, first, suggest a Quasi-Newton method in Infinite-dimensional Spaces (QNIS) in §2, which can be used to solve an operator equation that is governed by a non-linear operator mapping sets in a Hilbert space into another Hilbert space.

Next, we prove in §3 that the sequence $\{q_n\}$ generated by the QNIS procedure converges to the root of the operator equation if the later exists and the Fréchet derivative of the governing operator is invertible. In §4 we, first, give a proof to show that a finite-dimensional, approximate equation has a root if the original equation does, and then prove that the roots of finite-dimensional approximate equations converge to the root of the original operator equation under proper conditions. Finally, apply the above-mentioned algorithm to an inverse problem for parabolic differential equation, which shows that QNIS is efficient.

There are a lot of papers dealing with computation of inverse problems. We only list a few of them according the methods used as follows:

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1. The gradient or conjugate gradient methods[6,20,19,21], which need computing the derivative maps of the operators described by partial differential equations;
2. The generalized pulse spectrum technique (GPST)[7−8,25];
3. The finite-dimensional approximate modal methods[1−3];
4. The regularization methods[22−24];
5. The sequential quadratic programming (SQP) methods[16].

Finally, it should be pointed out that a superlinear rate of convergence in infinite-dimensional spaces is not trivial as it does in finite-dimensional spaces. [10] showed out that Q-superlinear convergence for a Lipchitzian operator $F(q)$ can be achieved if an initial operator A_0 is close to $F'(q^*)$ up to an arbitrary compact perturbation.

By the way, QNIS presented in the paper can also be applied to inverse problems in other PDEs.

2. A Quasi-Newton Method in Infinite-dimensional Spaces

We consider an operator equation

$$\Phi(q,u)=0, \tag{1}$$

where $\Phi \in C(Q\times U,\mathcal{F})$, $u\in U$ is a state of the system, $q\in Q$ is a parameter, U is a state space. Q is a topological space, U and $\mathcal{F}$ are Banach spaces, $C(Q\times U,\mathcal{F})$ is the set of all continuous maps on $Q\times U$ to $\mathcal{F}$.

We assume that (1) is well-posed, that is, $\forall q\in Q$ there is a unique $u\in U$ satisfying (1), and u depends continuously on q, then denote $u=u(q)$.

The inverse problem we address is to determine the pair (q,u) satisfying (1) and

$$\mathcal{M}u=z, \tag{2}$$

where $z\in Y$ is given, $\mathcal{M}: U\to Y$ is a given measurement operator.

The operator equations studied in the paper consist of partial differential equations and additional initial and/or boundary-value conditions.

For example, (1) is described by the following initial-boundary value problem for a parabolic equation:

$$\begin{aligned}&u_t=(q(x,y)u_x)_x+(q(x,y)u_y)_y+f(x,y,t), \quad (x,y)\in\Omega, \quad t\in(0,T)\\&\partial_\nu u\mid_{\partial\Omega}=0, \quad u(x,0)=u_0(x),\end{aligned} \tag{3}$$

which governs the temperature distribution in a nonhomogeneous isotropic solid or the pressure distribution in a fluid-containing porous medium. It is well-known that $\forall\, q\in L^\infty(\Omega)$ with $q(x)\ge c_0>0$, a.e. Ω, the problem (3) is well-posed and $U=H^1(\Omega\times(0,T))$.

The inverse problem considered is to determine $(q,u)\in Q\times U$ that satisfy (3) and

$$u\mid_{t=T}=z, \tag{4}$$

where $z\in H^1(\Omega)$ is given.