Journal of Computational Mathematics, Vol.16, No.4, 1998, 357–366.

QUADRILATERAL FINITE ELEMENTS FOR PLANAR LINEAR ELASTICITY PROBLEM WITH LARGE LAMÉ CONSTANT[*1)]

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Abstract

In this paper, we discuss the quadrilateral finite element approximation to the two-dimensional linear elasticity problem associated with a homogeneous isotropic elastic material. The optimal convergence of the finite element method is proved for both the L^2-norm and energy-norm, and in particular, the convergence is uniform with respect to the Lamé constant λ. Also the performance of the scheme does not deteriorate as the material becomes nearly incompressible. Numerical experiments are given which are consistent with our theory.

Key words: Planar linear elasticity, optimal error estimates, large Lamé constant, locking phenomenon

1. Planar linear elasticity problem

The two-dimensional linear elasticity problem associated with a homogeneous isotropic elastic material with pure displacements can be modelled by the following elliptic boundary value problem:

$$-\mu\Delta\vec{u} - (\mu+\lambda)\nabla(\text{div } \vec{u}) = \vec{f}, \qquad \text{in } \Omega, \tag{1.1}$$

$$\vec{u} = \vec{0}, \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\Omega \subset \Re^2$ is an open and bounded domain, $\vec{u} = (u_1, u_2)$ the displacement, $\vec{f}(x)$ the body force, and λ, μ the Lamé constants. Different equivalent formulations of (1.1)–(1.2) can be found in [3, 4, 11].

It is well known that the convergence rate for the standard displacement method using continuous linear finite elements deteriorates as the Lamé constant λ becomes large, i.e., the elastic material is nearly incompressible. Many finite element methods of higher order have been proposed which work uniformly well for all λ, see [2, 3, 1, 14]. However, all these elements are required to satisfy the Babuska-Brezzi-Ladyzenskaja condition for saddle point problems.

* Received July 2, 1996.
1) The work of this author was supported by the Direct Grant of CUHK.

In this paper, we use the simplest finite elements, i.e. the bilinear elements, which do not satisfy the Babuska-Brezzi-Ladyzenskaja condition, for the above elasticity problem. The key technique here is to use the reduced integration to deal with the second term in the equation (1.1). We apply the finite element method studied in [5] and [8] and use a variant of the stability condition for some subspaces of the global finite element space. We are able to prove the optimal error estimates in both the energy-norm and L^2-norm uniformly with respect to the Lamé constant λ, thus the convergence rate does not deteriorate even for nearly incompressible material.

Before ending this section, we introduce some notation used in the paper. For any positive integer m, $H^m(\Omega)$ denotes the usual Sobolev space of all square integrable functions over Ω with square integrable derivatives of order up to m, and its norm and semi-norm are denoted by $\|\cdot\|_m$ and $|\cdot|_m$. $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ with its functions vanishing on the boundary $\partial\Omega$ (in the sense of trace). $L_0^2(\Omega)$ is the space of all square integrable functions over Ω with their mean values in Ω vanishing.

2. The Bilinear Element Method

We first consider a very simple and regular domain in this section, i.e. the domain Ω is a rectangle. But we shall show in Section 4 that the method addressed in this section can be naturally extended to more general domains which may be triangulated using quadrilateral elements. As usual, we assume the Lamé constants μ, λ are in the following ranges $0 < \mu_0 \le \mu \le \mu_1$, $0 < \lambda < \infty$.

By Green's formula, it is easy to derive the weak formulation of the system (1.1)–(1.2):

Problem (P). *Find $\vec{u} \in [H_0^1(\Omega)]^2$ such that*

$$\mu(\nabla\vec{u}, \nabla\vec{v}) + (\mu+\lambda)(\text{div }\vec{u}, \text{div }\vec{v}) = (\vec{f}, \vec{v}), \quad \forall \vec{v} \in [H_0^1(\Omega)]^2, \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ or $[L^2(\Omega)]^2$.

Let $\mathcal{T}^h$ be a triangulation of the domain Ω into rectangular elements of mesh size h, which is obtained by refining a coarse rectangular mesh by dividing each coarse element into four subelements by linking the mid-points of the opposite edges of the coarse element. We then define the bilinear finite element space V_h by

$$V_h = \{\vec{v}_h \in [H_0^1(\Omega)]^2 : \ \vec{v}_{h|_K} \in [Q_1(K)]^2, \ \forall K \in \mathcal{T}^h \ \}, \tag{2.2}$$

where $Q_l(K)$ (l positive integer) is the space of polynomials of degree less than or equal to l in each variable on K.

Then the finite element problem to Problem (P) is formulated as follows:

Problem (P$_h$). *Find $\vec{u}_h \in V_h$ such that*

$$\mu(\nabla\vec{u}_h, \nabla\vec{v}_h) + (\mu+\lambda)I_1(\text{div }\vec{u}_h, \text{div }\vec{v}_h) = (\vec{f}, \vec{v}_h), \quad \forall \vec{v}_h \in V_h, \tag{2.3}$$

where $I_1(\cdot,\cdot)$ denotes the one-point Gaussian quadrature in each element, i.e.,

$$I_1(\text{div}\vec{u}_h, \text{div}\vec{v}_h) = \sum_{K\in\mathcal{T}^h} |K| \,\text{div }\vec{u}_h(q_K)\ \text{div }\vec{v}_h(q_K)$$