Journal of Computational Mathematics, Vol.16, No.4, 1998, 367–374.

THE MULTI-PARAMETERS OVERRELAXATION METHOD*1)

Wen Li
(*Dept. of Math. South China Normal Univ. Guangzhou*)
Zhao-yong You
(*Research Center for Applied Math. Xi'an Jiaotong Univ. Xi'an*)

Abstract

In this paper, first, we present the comparison theorem and the (generalized) Stein-Rosenberg theorem for the GMPOR method, which improves some recent results[9,11,13]. Second, we also give the convergent theorem of the GMPOR method, which generalizes the corresponding result of [9]. Finally, we provide the real interval such that the generalized extrapolated Jacobi iterative method and the generalized SOR methods simultaneously converge, one of the main results in [1] is extended.

Key words: GMPOR iterative method, convergence, comparison theorem, Stein-Rosenberg theorem

1. Introduction

Recently, many mathematical literatures have provided some new iterative methods for solving the linear system. Kuang[2] presented a two-parameter iterative method called TOR method, which is effective to give the numerical solution of partial differential equations. Wang[10] extended the TOR method to the GTOR methed and improves some results of [3, 11, 12]. In [5], Li also discussed the GTOR method, and extended the corresponding results of [10, 11]. Recently, Song and Dai[9] presented the multi-parameters overrelaxation (MPOR) method, whose specific cases involve the iterative methods mentioned as above. Now, let us make a generalization of the MPOR method.

Let
$$Ax = u, \tag{1.1}$$
where $A = D - \sum_{i=1}^{k} E_i - F$, and D is a nonsingular matrix. Then the generalized multi-parameters overrelaxation (GMPOR) method can be defined by
$$x^{m+1} = L(a_1, \cdots, a_k; b)x^m + v, \quad m = 0, 1, \cdots, \tag{1.2}$$
where x^0 is an initial approximation,
$$v = \Big(D - \sum_{i=1}^{k} a_i E_i\Big)^{-1} bu$$

* Received July 10, 1995.
1)Project supported by NNSF of China and NSF of Guangdong Province.

and

$$L(a_1,\cdots,a_k;b)=\Big(D-\sum_{i=1}^{k}a_iE_i\Big)^{-1}\Big[(1-b)D+\sum_{i=1}^{k}(b-a_i)E_i+bF\Big], \tag{1.3}$$

which is called the GMPOR iteration matrix, where a_i $i=1,\cdots,k$ and b are independent parameters, D is nonsingular matrix, E_i, $i=1,\cdots,k$ and F are any matrix (In [9] D, E and F respectively nonsingular block diagonal, strictly lower and upper triangular matrices).

Notice that for specific value of the parameters a_i and b, the GMPOR method reduces to the following well-known methods:

$L(0;1)=L_{GJ}$, the iteration matrix of the GJ method (generalized Jacobi method);

$L(1;1)=L_{GGS}$, the iteration matrix of the GGS method (generalized Gauss-Seidel method);

$L(0;b)=L_{GJOR}$, the iteration matrix of the GJOR method (generalized extrapolated Jacobi method);

$L(b;b)=L_{GSOR}$, the iteration matrix of the GSOR method (generalized SOR method);

$L(a;b)=L_{GAOR}$, the iteration matrix of the GAOR method (generalized AOR method).

$L(a_1,a_2;b)=L_{GTOR}$, the iteration matrix of the GTOR method (generalized TOR method).

From the above statement, one can easily understand that the GMPOR method includes the GJ method, GGS method, GJOR method, GSOR method, GAOR method, GTOR method and MPOR method as its specific cases. This paper is organized as follows. In Section 2, we present a comparison theorem and the (generalized) Stein-Rosenberg theorem for the GMPOR method, which improves some recent results[9,11]. Section 3 contains the convergence theorem of the GMPOR method for solving the nonsingular linear system, which extends the corresponding result of [9]. In the final Section, two theorems are given. The first theorem provides a necessary and sufficient condition such that the GMPOR method for solving the singular linear system is convergent. The second theorem reveals the real interval for which the GJOR method and the GSOR method are simultaneously convergent, one of the main results in [1] is generalized. All definitions and notations here are standard and can be found in [8] or [13].

2. Comparison Theorem and Stein-Rosenberg Theorem

Let n be a natural number. By $<n>$ we denote the set $\{1,\cdots,n\}$

Throughout this section we always assume that the following conditions hold:

$$L_i=D^{-1}E_i\geq 0,\ i=1,\cdots,k,\ U=D^{-1}F\geq 0,\quad \text{and } B=\sum_{i=1}^{k}L_i+U=L_{GJ} \tag{2.1}$$

$$\rho\Big(\sum_{i=1}^{k}L_i\Big)<1 \tag{2.2}$$