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COMPUTATION OF HOPF BRANCHES BIFURCATING FROM A HOPF/PITCHFORK POINT FOR PROBLEMS WITH Z_2–SYMMETRY[*1)]

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Abstract

This paper is concerned with the computation of Hopf branches emanating from a Hopf/Pitchfork point in a two-parameter nonlinear problem satisfying a Z_2-symmetry condition. Our aim is to present a new approach to the theoretical and computational analysis of the bifurcating Hopf branches at this singular point by using the system designed to calculate Hopf points and exploring its symmetry. It is shown that a Hopf/Pitchfork point is a pitchfork bifurcation point in the system. Hence standard continuation and branch-switching can be used to compute these Hopf branches. In addition, an effect method based on the extended system of the singular points is developed for the computation of branch of secondary (non-symmetric) Hopf points. The implementation of Newton's method for solving the extended system is also discussed. A numerical example is given.

Key words: Hopf/pitchfork point, Z_2-symmetry, Hopf point, bifurcation, Extended system

1. Introduction

This paper is devoted to the calculation of branches of Hopf points which emanate from a certain singular point of a two parameter nonlinear system

$$g(x, \lambda, \alpha) = 0 \quad g : X \times R \times R \to X \tag{1.1}$$

where X is a real Hilbert space, λ a bifurcation parameter, α an additional control parameter, and g is a smooth mapping. We assume

(H1) g is Z_2-symmetric: there exists a linear operator $s : X \to X$ satisfying (I: identical operator in X)

$$s \neq I, s^2 = I, sg(x, \lambda, \alpha) = g(sx, \lambda, \alpha) \text{ for } (x, \lambda, \alpha) \in X \times R^2. \tag{1.2}$$

It is well known that (1.2) induces the splitting

$$X = X_s \oplus X_a, \tag{1.3a}$$

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$$X_s := \{x \in X, sx = x\}, \quad X_a := \{x \in X, sx = -x\}. \tag{1.3b}$$

We say that x is symmetric if $x \in X_s$, and antisymmetric if $x \in X_a$.

Equation (1.1) is often studied as a first step towards the understanding of the evolution equation

$$\frac{dx}{dt} = g(x, \lambda, \alpha). \tag{1.4}$$

In particular, the transition from steady-state solutions to periodic solutions in (1.4) typically occurs at a Hopf point. Such a point is usually recognized by the occurrence of a pair of purely imaginary eigenvalues of g_x, the linearization of g with respect to x of the steady-state equation (1.1).

A Hopf/Pitchfork point (HP-point) $(x_0, \lambda_0, \alpha_0)$ is defined as a solution with $x_0 \in X_s$ of (1.1) where $g_x^0 := g_x(x_0, \lambda_0, \alpha_0)$ has a pair of simple pure imaginary eigenvalues, and a simple zero eigenvalue with an antisymmetric eigenvector. In the case of $X = R^n$ and $g(0, \lambda, \alpha) = 0$ for all λ and α, the coalescence of pitchfork and Hopf bifurcation points for a two-parameter system with Z_2-symmetry has been investigated by Langford and Iooss[8], Guckenheimer and Holmes[5] using Birkhoff normal form, respectively. They found interesting secondary Hopf bifurcation (Hopf bifurcation on the bifurcating non-symmetric steady-state solution branch) and, in addition, aperiodic (chaotic) motion from a periodic orbit in a neighborhood of the degenerate point.

In this paper, we will contribute to the analysis and the computation of Hopf points near a HP-point, which forms a foundation for understanding the complex dynamics of (1.4). The main analytical and numerical tool will be the following extended system (1.5a, b) for Hopf points which was given by Jepson[6], Griewank and Reddien[3]; here α will be used as a bifurcation parameter:

$$G(x, \phi_1, \phi_2, \lambda, \beta, \alpha) := \left\{ \begin{array}{c} g(x, \lambda, \alpha) \\ g_x(x, \lambda, \alpha)\phi_1 - \beta\phi_2 \\ g_x(x, \lambda, \alpha)\phi_2 + \beta\phi_1 \\ \langle l, \phi_1 \rangle - 1 \\ \langle l, \phi_2 \rangle \end{array} \right\} = 0, \tag{1.5a}$$

$$G : X \times X \times X \times R \times R \times R \to X \times X \times X \times R \times R := Y. \tag{1.5b}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in X and $l \in X$ is a normalizing vector. The system has already been used to study bifurcation phenomena near a Takens-Bogdanov point in the cases where symmetry is not broken (Spence, Cliffe and Jepson [10]) and where symmetry is broken (Wu, Spence and Cliffe[15]), respectively. Following the preliminaries of Section 2, we devote ourself to a straightforward analysis of the fact that, under some non-degenerate condition, a HP-point is a pitchfork bifurcation point in the system $G = 0$ with respect to α, see Section 3. The analysis relies on the result on symmetry breaking pitchfork bifurcation in Werner and Spence[12] by defining a symmetry of (1.5) similar to (1.2). One theoretical by-product of the analysis is that it gives a self-contained proof for the existence of the branch of secondary (non-symmetric) Hopf points emanating from a HP-point. This is also useful for numerical computations -for Hopf points on the symmetric and on the non-symmetric steady-state branches as