Journal of Computational Mathematics, Vol.16, No.5, 1998, 437–444.

THE PHYSICAL ENTROPY OF SINGLE CONSERVATION LAWS*1)

Gong-yan Lei
(*Department of Mathematics, Peking University, Beijing 100871, China*)

Abstract

By means of the comparisons with the formulas in statistical mechanics and thermodynamics, in this paper it is demonstrated that for the single conservation law $\partial_t u + \partial_x f(u) = 0$, if the flux function $f(u)$ is convex (or concave), then, the physical entropy is $S = -f(u)$; Furthermore, if we assume this result can be generalized to any $f(u)$ with two order continuous derivative, from the thermodynamical principle that the local entropy production must be non-negative, one entropy inequality is derived, by which the O.A. Olejnik's famous E- condition can be explained successfully in physics.

Key words: Conservation laws, Entropy, Entropy production.

1. Introduction

As the simplest representation of general conservation laws, the single conservation laws with one space variable have been thoroughly discussed.In the cases that the flux function is convex or concave, P.D. Lax obtained a general expression of the solutions for Cauchy problems[2]; in order to guarantee the uniqueness of the solution, O.A. Olejnik presented her famous E-condition which can be applied to general conservation laws with one space variable[4]; The Lax's concepts of entropy functions and related inequality also can be used in this special case[3]. In fact, both for differential equation's theory and for numerical methods, the single conservation laws are the primitive discussed objects.

However, even for the single conservation laws, there are still something not thoroughly clear, the problems manifest especially when the entropy and entropy condition are concerned, which are related to the uniqueness of the generalized solution. According to the P.D. Lax's definition, there are a lot of entropy functions for a single conservation law, the entropy inequality must be satisfied for every convex entropy; however, at least for some equations, it is reasonable to expect that there exists a "physical entropy", which determines the uniqueness of the generalized solution. Thus a question has arisen: what is the physical entropy and what is the corresponding entropy condition. In addition, the Olejnik's E- condition has clear and definite geometrical meaning, and by use of the Lax's entropy inequalities with suitable entropy

* Received September 26, 1995.
1)The Project Supported by National Natural Science Foundation of China

functions, or by the Kružkov's theorem , the Olejnik's E-condition can be derived, but what is its physical significance, especially, how does it relate to the physical entropy.

In this paper we try to answer these questions. First, in the cases that the flux functions are convex (or concave), beginning from the Lax's general expressions of solutions for Cauchy problems, by comparisons with formulas in the thermodynamic and statistical mechanics, it is demonstrated that the negative flux function can be regarded as the physical entropy; Furthermore, it is assumed that the above conclusion still holds for the general single conservation laws, by means of the thermodynamic principle that the local entropy production must be non-negative, an entropy inequality different from the Lax's one is derived, from which the Olejnik's E- condition can be explained successfully in physics. Furthermore, it is demonstrated that in a sense a strong discontinuity of a single conservation law can be considered as a most simple mathematical model of the non-equilibrium phase transitions.

In [6], F. Rezakhanlou has studied the hydrodynamic behavior of certain stochastic particle systems, and proved that under Euler scaling, the microscopic particle density converges to a determinate limit that is characterized as the entropy solution of a nonlinear conservation law; Rezakhanlou's results are very interesting, and what used by him is the statistical mechanics method; However, in [6] the entropy condition is not derived from the micromechanism, it just be proved. As compared with [6], the method applied in the present paper can be considered as a thermodynamic one, it does not concern with the micromechanism, but its results reveal that the mathematical concept of entropy functions should have more direct physical origin.

In section 2 some preliminary knowledges about the single conservation laws, statistical mechanics and thermodynamics, which are necessary for the present paper, are briefly introduced. Section 3 contains our main results. Finally in section 4 there are some concise discussions.

2. Preliminary

The Cauchy problems for single conservation laws with one space variable can be expressed as follows[2,7]:

$$u_t + f(u)_x = 0, \quad -\infty < x < \infty, \, t > 0 \tag{2.1}$$

$$u(x,0) = \phi(x), \quad -\infty < x < \infty \tag{2.2}$$

where x and t are the independent variables, $u(x,t)$ is the unknown function, $f(u)$ is called flux. If for all smooth test functions $w(x,t)$ which vanish for $|x|+t$ large enough, the function $u(x,t)$ and $f(u)$ are integrable, and satisfy the following relation:

$$\int_0^\infty \int_{-\infty}^\infty [w_t u + w_x f] dx dt + \int_{-\infty}^\infty w(x,0)\phi(x) dx = 0 \tag{2.3}$$

then, function $u(x,t)$ is defined as a weak solution of (2.1) (2.2), as is well known for piecewise continuous solutions, (2.3) is equivalent to the Rankine-Hugoniot jump condition

$$s[u] - [f] = 0 \tag{2.4}$$