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# PROBABILISTIC ANALYSIS OF GALERKIN-LIKE METHODS FOR THE FREDHOLM EQUATION OF THE SECOND KIND[*1)]

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**Abstract**

This paper deals with the approximate solution of the Fredholm equation $u - T_K u = f$ of the second kind from a probabilistic point of view. With Wiener type measures on the set of kernels and free terms we determine statistical features of the approximation process, i.e., the most likely rate of convergence and the dominating individual behavior. The analysis carried out for a kind of Galerkin-like method.

*Key words*: Probabilistic analysis, fredholm equation, galerkin-like method, abstract wiener space

## 1. Introduction

Quantitative probabilistic analysis was carried out for several numerical problems. For a systematic survey, we refer to Traub et al. (1988) and references therein. Smale (1985) gave the first quantitative analysis for concrete measure. He expected that the approach there might lead to a more systematic way of analysing for the cost of numerical algorithms. Heinrich (1991) continued this line and gave the first quantitative analysis for concrete measures and algorithms for integral equation of the second kind. There, the analysis was carried out for the Galerkin method and the iterated Galerkin method. It is natural to ask whether other numerical problems can be analyzed from this point of view. In this paper we get counterparts for a kind of Galerkin-like method, which was proposed by Schock (1971). For brevity, later on, it was called Q-method (see, e.g., Schock (1982)). For a more precise discussion of relation between Q-method and Galerkin method and iterated Galerkin method we refer to Schock (1982).

Finally, we briefly outline the contents of this paper. Section 2 reviews some basic facts about Gaussian measures. Section 3 deals with the main problem in terms of general Banach spaces and Gaussian measures. Section 4 specifies our main problem and formulates the principal results. Section 5 and 6 are devoted to the proofs of the principal results.

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## 2. Preliminaries on Gaussian Measures

We consider only Banach spaces over the field of reals throughout this paper. Given Banach spaces $X$ and $Y$ we let $L(X,Y)$ denote the spaces of all bounded linear operators $T$ from $X$ to $Y$, equipped with the operator norm $\|T\|$. $K(X,Y)$ is the space of compact operators, and we write $L(X)$ and $K(X)$ if $X = Y$. $X^*$ stands for the dual space of $X, \mathcal{B}(X)$ is the $\sigma$-algebra of all Borel subsets of $X$. The symbol $\langle , \rangle$ is used for the duality between $X$ and $X^*$, while $( , )$ always denotes inner products. If $X = H$ is a Hilbert space, we identify $X^*$ with H in the usual way, so that $\langle , \rangle$ and $( , )$ coincide. For $x^* \in X^*$, $y \in Y$, $x^* \otimes y \in L(X,Y)$ denotes the operator defined by $(x^* \otimes y)(x) = \langle x, x^* \rangle y$.

Now we list some basic notions and facts about Gassian measures, the emphasis laid on the operator theoretic aspect. A Gaussian measure on a Banach space $X$ is a Radon probability measure $\mu$ such that each $x^* \in X^*$ is a symmetric Gaussian random variable on $(X,\mu)$ (which may be degenerate, that is, $= 0$ almost everywhere). We shall consider only symmetric, i.e., mean zero Gaussian measures. For a Hilbert space $H$ we let $\gamma_H$ denote the standard Gaussian cylindrical probability (see [Kuo (1975)], [Pietsch (1980)]). For $T \in L(H,X)$ let

$$E_\gamma(T) = \sup_{\substack{F \subset H \\ \dim F < \infty}} \int_F \|Th\| d\gamma_F(h), \tag{1}$$

and let $\Pi_\gamma(H,X)$ denote the set of all $T \in L(H,X)$ with $E_\gamma(T) < \infty$. $E_\gamma$ is a norm on $\Pi_\gamma(H,X)$ turning it into a Banach space. It is easily checked that

$$\|T\| \leq (\pi/2)^{1/2} E_\gamma(T). \tag{2}$$

For a further Hilbert space $H_0$, a Banach space $X_0$, $S \in L(H_0,H)$ and $U \in L(X,X_0)$,

$$E_\gamma(UTS) \leq \|U\| E_\gamma(T) \|S\|, \tag{3}$$

(it follows from [Linde and Pietsch (1974), Lemma 2]). Let $R_\gamma(H,X)$ be the closure of the finite rank operators in $\Pi_\gamma(H,X)$. For $T \in L(H,X)$, let $T_{\gamma_H}$ denote the cylindrical probability measure induced on $X$ by $T$, that is, $T_{\gamma_H} = \gamma_H(T^{-1}(B))$ for cylindrical sets $B$. Now $T \in R_\gamma(H,X)$ if and only if $T_{\gamma_H}$ has an extension $\tilde{T}_{\gamma_H}$ to $\mathcal{B}(X)$ which is a radon measure (such an extension is unique). So $T \in R_\gamma(H,X)$ implies that $\tilde{T}_{\gamma_H}$ is Gaussian. Conversely, If $\mu$ is a Gaussian measure on $X$, there is a separable Hilbert space $H$ and an injection $J \in R_\gamma(H,X)$ with $\mu = \tilde{J}_{\gamma_H}$. $H$ and $J$ are essentially unique (up to isometries). Note that $(J,H,X)$ is then an abstract Wiener space (see [Kuo(1975)]). If $\mu = \tilde{T}_{\gamma_H}$, $T \in R_\gamma(H,X)$, then $C_\mu = TT^*$ is the covariance operator of $\mu$, the closure of $\text{Im}T$ is the support of $\mu$, and

$$E_\gamma(T) = \int_X \|x\| d\mu(x). \tag{4}$$

These facts can be found in [Kuo (1975), Linde et al (1974), Traub et al (1988)]. If $X = G$ is a Hilbert space, then $R_\gamma(H,G)$ coincides with the class of Hilbert-Schmidt