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BOUNDARY PENALTY FINITE ELEMENT METHODS FOR BLENDING SURFACES, I BASIC THEORY*[1)]

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Abstract

When parametric functions are used to blend 3D surfaces, geometric continuity of displacements and derivatives until to the surface boundary must be satisfied. By the traditional blending techniques, however, arbitrariness of the solutions arises to cause a difficulty in choosing a suitable blending surface. Hence to explore new blending techniques is necessary to construct good surfaces so as to satisfy engineering requirements. In this paper, a blending surface is described as a flexibly elastic plate both in partial differential equations and in their variational equations, thus to lead to a unique solution in a sense of the minimal global surface curvature. Boundary penalty finite element methods (BP-FEMs) with and without approximate integration are proposed to handle the complicated constraints along the blending boundary. Not only have the optimal convergence rate $O(h^2)$ of second order generalized derivatives of the solutions in the solution domain been obtained, but also the high convergence rate $O(h^4)$ of the tangent boundary condition of the solutions can be achieved, where h is the maximal boundary length of rectangular elements used. Moreover, useful guidance in computation is discovered to deal with interpolation and approximation in the boundary penalty integrals. A numerical example is also provided to verify perfectly the main theoretical analysis made. This paper yields a framework of mathematical modelling, numerical techniques and error analysis to the general and complicated blending problems.

Key words: Blending surfaces, parametric surfaces, plate, mathematical modelling, variational equations, finite element methods, boundary penalty method, computer geometric aided design

1. Introduction

Blending surfaces is said if when two frame surfaces (or bodies) are located already, a smoothly transferring surface is sought to connect the two frame surfaces along certain boundary. Usually, the terminology "smoothness" means that the blending surface belongs to geometric continuity C^1 (Foley et al. (90) [14), i.e., the blending surface and its tangent plane are continuous until the joint boundary. Many literatures are reported on this subject. We merely mention a few of them relevant to this paper. Uniform

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algebraic polynomials of low order have been discussed in Hoffmann and Hopcroft(92), and Ohkura and Kakazu(92), to blend simple frames, such as those with quadratic and cubic surfaces. Piecewise spline functions can also be used to obtain rather complicated blending surfaces (see Kosters(89), Bajaj and Ihn(92)). This paper is, however, devoted to find efficient approaches to construct good surfaces to blend general, complicated frame-surfaces (or frame-bodies), which may be used in airplane, ships, grand buildings, and astronautic shuttle-station. The existing blending techniques become awkward in handling arbitrary joining locations and boundary conditions. To manages complicated blending, we solicit partial differential equations (PDEs) of order four describing elastic plates, and seek additional conditions of unique solutions. Note that techniques using PDEs are given in Bloor and Wilson(90,91) but still to deal with simple cases.

A plate in algebraic functions, e.g., $z = f(x, y)$ may serve well as a blending surface, applicable to simple surface modelling. However, parametric functions are more advantageous to represent general and complicated 3D surfaces. When the blending surfaces are connected to the frame boundary satisfying the displacement and tangent conditions, there occur multiple parametric surfaces, unfortunately. A simple case in 2D blending curves is illustrated in Foly et. al (90, p.486) [14]. There arises a question how to choose a suitable, unique blending surface. This is important to computer aided design. As far as our current knowledge (referring to Choi(91), Farin(90), Fisher(94), Koenderink(90), Nutbourne and Martin(89), Su and Liu(89), Warren(89), as well as the recent fourth SIAM Conference on Geometric Design, Nashville, Tennessee, Nov., 6-9, 1995, it seems to exist no literatures to address this problem. This paper is, therefore, intended to study such a challenging topic.

The organism of this paper is as follows. In the next section, mathematical modelling of blending surfaces is given with PDEs and their variational forms, thus to yield unique solutions. In Section 3, three kinds of boundary penalty finite element methods (BP-FEMs) are presented, to simplify the algorithms involving the complicated boundary conditions. Error analysis is then made in Section 4, accompanied with comparison; a simple numerical example is given in Section 5 to verify the optimal convergence rates.

2. Mathematical Modelling of Blending Surfaces

Consider that a surface is sought to join two given frame bodies V_1 and V_2 at the left boundary ∂V_1 and the right boundary ∂V_2. Suppose that ∂V_1 and ∂V_2 are disjointed to each other (see Fig.1). Since the algebraic function $y = f(x, y)$ is difficult to represent the closed surface shown in Fig.1, we solicit parametric functions instead. Choose two parameters r and t in a unit solution area $\Omega\{(r,t),\ 0 < r < 1,\ 0 < t < 1\}$, and use three parametric functions

$$x = x(r,t), \quad y = y(r,t), \quad z = z(r,t), \quad (r,t) \in \Omega \tag{2.1}$$

to represent the blending surface in Fig.1. Naturally, we denote the left boundary $\partial\Omega|_{r=0}$, and the right boundary $\partial\Omega|_{r=1}$, to represent ∂V_1 and ∂V_2, respectively. Denote the boundary of Ω as (see Fig.2) by $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \overline{AB} \cup \overline{CD}$, $\Gamma_2 = \overline{AC} \cup \overline{BD}$,