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THE CALCULUS OF GENERATING FUNCTIONS AND THE FORMAL ENERGY FOR HAMILTONIAN ALGORITHMS*1)

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Abstract

In [2–4], symplectic schemes of arbitrary order are constructed by generating functions. However the construction of generating functions is dependent on the chosen coordinates. One would like to know that under what circumstance the construction of generating functions will be independent of the coordinates. The generating functions are deeply associated with the conservation laws, so it is important to study their properties and computations. This paper will begin with the study of Darboux transformation, then in section 2, a normalization Darboux transformation will be defined naturally. Every symplectic scheme which is constructed from Darboux transformation and compatible with the Hamiltonian equation will satisfy this normalization condition. In section 3, we will study transformation properties of generator maps and generating functions. Section 4 will be devoted to the study of the relationship between the invariance of generating functions and the generator maps. In section 5, formal symplectic erengy of symplectic schemes are presented.

Key words: Generating function, calculus of generating functions, Darboux transformation cotangent bundles, Lagrangian submanifold, invariance of generating function, formal energy.

1. Darboux Transformation

Consider cotangent bundle $T^*R^n \simeq R^{2n}$ with natural symplectic structure

$$J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \tag{1.1}$$

and the product of cotangent bundles $(T^*R^n) \times (T^*R^n) \simeq R^{4n}$ with natural product symplectic structure

$$\tilde{J}_{4n} = \begin{bmatrix} -J_{2n} & 0 \\ 0 & J_{2n} \end{bmatrix}. \tag{1.2}$$

Correspondingly, we consider the product space $R^n \times R^n \simeq R^{2n}$. Its cotangent bundle, $T^*(R^n \times R^n) = T^*R^{2n} \simeq R^{4n}$ has natural symplectic structure

$$J_{4n} = \begin{bmatrix} 0 & I_{2n} \\ -I_{2n} & 0 \end{bmatrix}. \tag{1.3}$$

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Choose symplectic coordinates $z=(p,q)$ on the symplectic manifold, then for symplectic transformation $g: T^*R^n \longrightarrow T^*R^n$, we have

$$\text{gra}(g)=\left\{\begin{bmatrix} g_z \\ z \end{bmatrix}, \quad z\in T^*R^n\right\}, \tag{1.4}$$

it is a Lagrangian submanifold of $T^*R^n\times T^*R^n$ in $\tilde{R}^{4n}=(R^{4n},\tilde{J}_{4n})$. Note that on R^{4n} there is a standard symplectic structure (R^{4n},J_{4n}). A generating map

$$\alpha: T^*R^n\times T^*R^n \longrightarrow T^*(R^n\times R^n)$$

maps the symplectic structure (1.2) to the standard one (1.3). In particular, α maps Lagrangian submanifolds in $(R^{4n},\tilde{J}_{4n})$ to Lagrangian submanifolds L_g in (R^{4n},J_{4n}). Suppose that α satisfies the transversality condition of g, then

$$L_g=\left\{\begin{bmatrix} d\phi_g(w) \\ w \end{bmatrix}, w\in R^{2n}\right\}. \tag{1.5}$$

ϕ_g is called generating function of g. We call this generating map α (linear case) or α_* (nonlinear case) Darboux transformation, in other words, we have the following definition.

Definition 1.1. A linear map

$$\alpha=\begin{bmatrix} A_\alpha & B_\alpha \\ C_\alpha & D_\alpha \end{bmatrix}, \tag{1.6}$$

which acts as the followings

$$\begin{bmatrix} z_0 \\ z_1 \end{bmatrix}\in R^{4n} \longmapsto \alpha\begin{bmatrix} z_0 \\ z_1 \end{bmatrix}=\begin{bmatrix} A_\alpha z_0+B_\alpha z_1 \\ C_\alpha z_0+D_\alpha z_1 \end{bmatrix}=\begin{bmatrix} w_0 \\ w_1 \end{bmatrix}\in R^{4n}$$

is called a Darboux transformation, if

$$\alpha' J_{4n}\alpha=\tilde{J}_{4n}. \tag{1.7}$$

Denote

$$E_\alpha=C_\alpha+D_\alpha, \quad F_\alpha=A_\alpha+B_\alpha. \tag{1.8}$$

We have

Definition 1.2.

$$Sp(\tilde{J}_{4n},J_{4n})=\{\alpha\in GL(4n)|\alpha' J_{4n}\alpha=\tilde{J}_{4n}\}=Sp(\tilde{J},J);$$
$$Sp(J_{4n})=\{\beta\in GL(4n)|\beta' J_{4n}\beta=J_{4n}\}=Sp(4n);$$
$$Sp(\tilde{J}_{4n})=\{\gamma\in GL(4n)|\gamma'\tilde{J}_{4n}\gamma=\tilde{J}_{4n}\}=\widetilde{Sp}(4n).$$

Definition 1.3. A special case of Darboux transformation $\alpha_0=\begin{bmatrix} J_{2n} & -J_{2n} \\ \frac{1}{2}I_{2n} & \frac{1}{2}I_{2n} \end{bmatrix}$ is called Poincare transformation.