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ON A THEOREM OF BERNSTEIN AND ITS APPLICATIONS TO WEIGHTED MINIMAX SERIES*

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Abstract

In this paper, some results about approximation in a norm S induced by the minimax series are studied. Then a Bernstein-type theorem for the norm S is established. Finally the Bernstein theorem is applied to prove the existence of certain equalities with minimax series and weighted minimax series.

Key words: Approximation theory, polynomials, Bernstein theorem, minimax series.

1. Introduction

Let f be a continuous function on $[a,b]$. Π_n will designate the set of all polynomials of degree less or equal than n and Π the set of all polynomials. As is well known, for each n the minimax of f is given by:

$$E_n(f) = \|f - p_n\|_\infty = \inf_{p\in\Pi_n} \|f-p\|_\infty,$$

where p_n is the best uniform approximation of f in Π_n.

Let us also consider the minimax series given by the expression

$$S(f) \equiv \sum_{k=0}^{\infty} E_k(f) \tag{1.1}$$

The set of functions for which $S^*(f) = \sum_{k=0}^{\infty} E_k^*(f) < \infty$, where $E_k^*(f)$ denotes the error of best approximation of $f \in C[0,2\pi]$ with trigonometric polynomials was already studied by S.N. Bernstein. He proved that such functions are of class $C^1[a,b]$.

The series (1.1) can be seen as a measuare of "how good" the funtion f can be approximated by polynomials, in the next sense. If f and $g \in C[a,b]$ and $\|f\|_\infty = \|g\|_\infty$ we will say that f is better approximated by polynomials than g on $[a,b]$ if and only if $S(f) < S(g)$.

On the other hand let $x_0 \in [a,b]$ be fixed. We set:

$$M_0 = \{f \in C[a,b] : f(x_0) = 0\},$$

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$$\Pi^0 = \{\text{polynomials } p : p(x_0) = 0\},$$
$$\Pi_n^0 = \{\text{polynomials } p \in \Pi_n : p(x_0) = 0\},$$

and

$$C_0 = \{f \in M_0 : S(f) < \infty\}.$$

By introducing,

$$S : C_0[a,b] \to \mathbb{R}$$
$$f \to S(f),$$

it can be proved that (C_0, S) is a normed space. Furthermore, $\forall f \in C[a,b]$ such that $S(f) < \infty$ there exists $g = f - f(x_0)$ such that $g \in C_0[a,b]$ and $S(f) = S(g)$. The approximation of a function $f \in (C_0, S)$ by polynomials in Π_n, is studied in [5].

(i) For a given $f \in C_0$, let $p_n \in \Pi_n$ be a best approximation of f in the norm S. Who is p_n?.

(ii) Is the space of all polynomials Π dense in (C_0, S)?.

The answer to these questions is contained in [5]. We recall in the next section some results proved in [5] in order to make this paper selfcontained. Also the convergence in the space (C_0, S) is analized in [5] and it is proved that it is a Banach space.

2. Approximation by Polynomials in the Space (C_0, S)

Let f be a function in (C_0, S). We consider the best approximation of f in Π_n $n = 0, 1, \cdots$ with respect to the norm S. That is, find $q_n \in \Pi_n$, such that:

$$S(f - q_n) = \inf_{p \in \Pi_n} S(f - p)$$

Let $p \in \Pi_n$. Then

$$E_k(f - p) = E_n(f), \quad (k \geq n)$$

and

$$E_k(f - p) \geq E_n(f), \quad (k < n).$$

Then,

$$\inf_{p \in \Pi_n} S(f - p) = \inf_{p \in \Pi_n} \sum_{k=0}^{n-1} E_k(f - p) + C(f),$$

where $C(f) = \sum_{k \geq n} E_k(f)$.

The existence of q_n can be deduced from the fact that (Π_n^0, S) is a normed space of finite dimension. Let us solve the following question, who is the approximant q_n?.

Proposition 1. *Let $f \in C[a,b]$. Then $S(f - q_n) = \inf_{p \in \Pi_n} S(f - p)$ iff there exists a constant C such that*

$$q_n = p_n + C \text{ where } \|f - p_n\|_\infty = \inf_{q \in \Pi_n} \|f - q\|_\infty \tag{2.1}$$