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TOTAL GENERALIZED MINIMUM BACKWARD ERROR ALGORITHM FOR SOLVING NONSYMMETRIC LINEAR SYSTEMS*1)

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Abstract

This paper extendes the results by E.M. Kasenally[7] on a Generalized Minimum Backward Error Algorithm for nonsymmetric linear systems $Ax = b$ to the problem in which pertubations are simultaneously permitted on A and b. The approach adopted by Kasenally has been to view the approximate solution as the exact solution to a perturbed linear system in which changes are permitted to the matrix A only. The new method introduced in this paper is a Krylov subspace iterative method which minimizes the norm of the perturbations to both the observation vector b and the data matrix A and has better performance than the Kasenally's method and the restarted GMRES method[12]. The minimization problem amounts to computing the smallest singular value and the corresponding right singular vector of a low-order upper-Hessenberg matrix. Theoratical properties of the algorithm are discussed and practical implementation issues are considered. The numerical examples are also given.

Key words: Nonsymmetric linear systems, Iterative methods, Backward error.

1. Introduction

An important aspect of any iterative method for approximating the solution of a linear system

$$Ax = b, \tag{1.1}$$

where A is an $n \times n$ real nonsymmetric matrix and b is an n-vector, is to decide at what point to stop the iteration. We customarily use the residual error as a stopping condition. The residual error $r_m = b - Ax_m$ can be viewed as a perturbation to the vector b such that the approximate solution is an exact solution of the perturbed linear system $Ax = b + \delta$, in which changes are permitted to the vector b only. The GMRES algorithm is based on classical Krylov subspace techniques and computes an approximate solution restricted to an affine space while minimising the backward perturbation norm of the vector b. From this backward error analysis of view E.M. Kasenally has viewed the approximate solution as an exact one of the perturbed linear system $(A - \Delta)x = b$,

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in which changes are permitted to the matrix A only. The Krylov subspace algorithm GMBACK proposed by Kasenally[7] computes an approximate solution restricted to an affine space while minimizing the backward perturbation norm of the matrix A. In this paper we view the approximate solution as an exact solution of the perturbed linear system $(A-\Delta)x = b+\delta$, in which changes are simultaneously permitted on matrix A and b[1,9,10]. A new Krylov subspace algorithm TGMBACK, which computes an approximate solution restricted to an affine space and minimizing the backward perturbation norm of the matrix A and vector b is presented. This minimization problem amounts to computing the smallest singular value and the corresponding right singular vector of a low-order upper Hessenberg matrix. The advantage for considering the algorithms which minimize the backward error is that there is often some uncertainty in the data A and b of the original linear systems and we can compare the backward error with the size of the uncertainty. Moreover, we found from numerical examples that the new method has better performance than Kasenally's method and restarted GMRES method.

The outline of this paper is as follows. Section 2 gives a backward error analysis for any iterative method for solving linear systems. The TGMBACK algorithm is introduced in Section 3. Some practical implementation issues and the numerical examples are presented in Section 4 and Section 5, respectively.

2. Backward Error Analysis for Iterative Methods

Consider the linear system in (1.1), where A is a large nonsymmetric matrix. Let $\{x_m\}$ be a sequence of approximate solutions produced by any iterative method. We first compare the residual error $r_m \equiv b - Ax_m$ with the minimum backward error $\Delta_{\min}$ in matrix A which satisfies $\|\Delta_{\min}\|_F = \min\{\|\Delta\|_F : (A-\Delta)x_m = b\}$.

Theorem 2.1. *Let x_m be an approximate solution of the linear system* (1.1) *and $\Delta_{\min}$ be the minimum backward error Δ in the matrix A such that $(A-\Delta)x_m = b$. Then*

$$\|\Delta_{\min}\|_F = \|r_m\|_2/\|x_m\|_2, \tag{2.1}$$

where $\|.\|_F$ is the Frobenious norm.

Proof. The residual equation

$$r_m = b - Ax_m$$

can be rewritten as follows

$$\Big(A + \frac{r_m x_m^T}{\|x_m\|_2^2}\Big)x_m = b$$

which implies that

$$\|\Delta_{\min}\|_F \le \|r_m x_m^T/\|x_m\|_2^2\|_F = \|r_m\|_2/\|x_m\|_2. \tag{2.2}$$

On the other hand, we have

$$(A-\Delta_{\min})x_m = b$$