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ITERATIVE METHODS WITH PRECONDITIONERS FOR INDEFINITE SYSTEMS*[1)]

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Abstract

For the sparse linear equations $Kx = b$, where K arising from optimization and discretization of some PDEs is symmetric and indefinite, it is shown that the $L\overline{L}^T$ factorization can be used to provide an "exact" preconditioner for SYMMLQ and UZAWA algorithms. "Inexact" preconditioner derived from approximate factorization is used in the numerical experiments.

Key words: Generalized condition number, Indefinite systems, Factorization method

1. Introduction

Symmetric indefinite systems of linear equations arise in many areas of scientific computation. In this paper, we will discuss the solution of sparse indefinite system of the form

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1}$$

where $A \in R^{n\times n}$ is a symmetric positive definite matrix, $B \in R^{m\times n}$ has full row rank $m \le n$, $C \in R^{m\times m}$ is symmetric positive semidefinte, $f \in R^n$ and $g \in R^m$. In this case, the linear equations has the unique solution[8−10]. For simplicity, we denote the equations as $Kx = b$.

Discretizations of the Stokes equations or other PDEs produce the linear equations as (1). In optimization, when barrier or interior-point methods are applied to some linear or nonlinear programs, the Karush-Kuhn-Tucker optimality conditions also lead to a set of equations as (1). The system often need not to be solved exactly, therefore it is appropriate to consider iterative methods and preconditioners for the indefinite matrix K.

Our main aim is to present a simple result that shows how to use the $L\overline{L}^T$ factorization of K[8] to construct a preconditioner for iterative methods. The iterative methods to be discussed are the Paige-Saunders algorithm named as SYMMLQ[7] and the UZAWA method[1].

The rest of the paper is organized as follows. In section 2, we derive the exact preconditioner from the $L\overline{L}^T$ factorization and take inexact preconditioner from approximate factorization into account. In section 3, two iterative methods, SYMMLQ

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and UZAWA algorithms with preconditioners, are presented. In section 4, we present the numerical results and show the effectiveness of the preconditioners.

2. Preconditioning Indefinite System Using $L\overline{L}^T$ Factorization

The indefinite system $Kx = b$ arising from optimization and PDEs is often ill-conditioned. It is appropriate to take a positive definite matrix $M = CC^T$ as preconditioner for K so that $C^{-1}KC^{-T}$ has lower condition number or better eigenvalue distribution.

The following theorem presents the $L\overline{L}^T$ factorization of K. For more detail, see [8].

Theorem 2.1. *Given any symmetric indefinite matrix*

$$K = \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}, \tag{2}$$

where A, B and C are the same as that defined in (1). Then we have

$$K = L\overline{L}^T, \tag{3}$$

$$L = \begin{pmatrix} l_{11} & \\ l_{21} & l_{22} \end{pmatrix}, \quad \overline{L}^T = \begin{pmatrix} l_{11}^T & l_{21}^T \\ & -l_{22}^T \end{pmatrix}, \tag{4}$$

where $l_{11} \in R^{n\times n}$ and $l_{22} \in R^{m\times m}$ are lower triangular matrices, $l_{21} \in R^{m\times n}$.

The matrices l_{11}, l_{21} and l_{22} can be easily calculated from the following matrix equations:

$$A = l_{11}l_{11}^T, \tag{5}$$

$$B = l_{21}l_{11}^T, \tag{6}$$

$$C + l_{21}l_{21}^T = l_{22}l_{22}^T. \tag{7}$$

If we take LL^T as the preconditioner of K, it is easily verified that

$$\overline{K} = L^{-1}KL^{-T} = \begin{pmatrix} I_{11} & \\ & -I_{22} \end{pmatrix} \equiv J, \tag{8}$$

where $I_{11} \in R^{n\times n}$ and $I_{22} \in R^{m\times m}$ are identity matrices. This means the "perfect" preconditioner for K is the matrix

$$M = LL^T, \tag{9}$$

since the preconditioned matrix $\overline{K}$ has at most two distinct eigenvalues and the Paige-Saunders algorithm converges in at most two iterations[2]. The matrix LL^T is named as the exact preconditioner for K.

In practice, we will use "inexact" preconditioner, which is derived from the $L\overline{L}^T$ factorization of an approximation to K. For the inexact preconditioner, we have the following results. Let $\lambda_{\max}(K)$ denote the maximum eigenvalue of K, $\lambda_{\min}(K)$ the minimum eigenvalue. $\lambda_1(K)$, $\lambda_2(K)$ is the maximum and minimum of $|\lambda(K)|$ respectively. The generalized condition number of K is defined by $\kappa(K) = |\lambda_1(K)/\lambda_2(K)|$.