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HIGH ACCURACY ANALYSIS OF THE WILSON ELEMENT*

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Abstract

In this paper, the Wilson nonconforming finite element is considered for solving a class of second-order elliptic boundary value problems. Based on an asymptotic error expansion for the Wilson finite element, the global superconvergences, the local superconvergences and the defect correction schemes are presented.

Key words: Finite elements, Defect correction, Global superconvergence, Wilson element.

1. Introduction

It is well known that superconvergence estimates and error expansions for the conforming finite elements are well studied in many papers. We refer to [16] for a survey on various results of superconvergence and to [10] for a fundamental work on asymptotic error expansions and to [1]–[3] for some techniques on high accuracy analysis. However, for the nonconforming elements, due to the reduced continuity of trial and test functions, it becomes more difficult to discuss superconvergence properties and related asymptotic error expansions. Naturally, people want to ask if the accuracy of the nonconforming element approximation can be improved by means of other methods. However, up to present, the work in this field have seldom been found in the literature. For the relatively simple Wilson element, a result of superconvergence in the energy norm has been obtained in [7] for a model situation and, within the same setting, independently, Chen and Li [8] have obtained L^p and $W^{1,p}$ $(1 \leq p \leq \infty)$ error estimates as well as the extrapolation results. For more general equation, Chen and Li[8] have obtained the error expansions and the pointwise superconvergence error estimates for the gradient. For the Carey nonconforming element, the superconvergence estimate of the gradient at the element centroid has been proved in [20]. However, these superconvergence results are only pointwise and particular. In order to get the high accuracy of the nonconforming elements as that for the conforming elements, we carefully analyse the Wilson element in this paper. We find that the Wilson element not only has the pointwise superconvergence, but also has the asymptotic error expansions, the global and local superconvergences, the defect corrections and the extrapolations. The key

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point of analysis is the expansions of some integral identities. And this kind of technical details can be found in [1], [4] and the original paper [10].

It is known that the nonconforming Wilson finite element passes the Irons patch test on general quasi-uniform quadrilateral meshes and the rate of convergence in the energy norm is of first order. It is shown by an example in [5] that this rate of convergence is optimal. Thus, in contrast to conforming quadratic finite element which achieves a second-order rate of convergence in the energy norm, the Wilson finite element loses one order of accuracy because of its nonconforming. In this paper, we present a method that as long as post-processing on the finite element solution, i.e., using a high order interpolation for the finite element solution, we have not only obtained the global superconvergences of a second order or higher order rate of convergence, but also have obtained the local superconvergence and the defect correction schemes.

2. Global Superconvergence

For simplicity, let Ω be the unit square in the xy-plane. We consider the following boundary value problem

$$\begin{cases} -Lu \equiv -\dfrac{\partial}{\partial x}\Big(A_1\dfrac{\partial u}{\partial x}\Big) - \dfrac{\partial}{\partial y}\Big(A_2\dfrac{\partial u}{\partial y}\Big) = f & \text{in} \quad \Omega, \\ u = 0 & \text{on}\, \partial\Omega, \end{cases} \tag{1}$$

where A_1, A_2 and f are sufficiently smooth functions defined on Ω and $A_1, A_2 \geq \alpha = const > 0$. Let $T^h = \{e_{ij}\}_{i,j=1}^{n,m} = \{e\}$ be a rectangular partition of the domain Ω, where n, m are two positive integers, $e_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ are rectangular elements, and

$$0 = x_0 \leq x_1 \leq \cdots, \leq x_n = 1,\ 0 = y_0 \leq y_1 \leq \cdots, \leq y_m = 1$$

are two one-dimensional partitions on the x-axis and y-axis, respectively. Define $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$, and the mesh size $h = \max\{h_i, k_j\}_{i,j=1}^{n,m}$. As usual T^h is said to be quasi-uniform if there exists a positive constant c such that

$$ch \leq \min\{h_i, k_j\}_{i,j=1}^{n,m}.$$

Furthermore, T^h is said to be unidirectionally uniform if

$$h_i = h_1,\ i = 1, \cdots, n, \text{ and } \quad k_j = k_1,\ j = 1, \cdots, m.$$

For the mesh T^h, let N_h denote the set of vertices and we define V^h to be the Wilson finite element space which consists of all functions $v \in L^2(\Omega)$ such that v is piecewise quadratic over Ω and continuous on N_h and v vanishes on $N_h \cap \partial\Omega$, i.e., six degrees of freedom on the element e of the Wilson element are uniquely determined by its values at four vertices of element e and two integrals $\int_e \frac{\partial^2 v}{\partial x^2} dxdy$ and $\int_e \frac{\partial^2 v}{\partial y^2} dxdy$. The Wilson finite element solution of the equation (1), $R_h u \in V^h$, is defined through the relation

$$a_h(R_h u, \varphi_h) = (f, \varphi_h), \quad \forall \varphi_h \in V^h, \tag{2}$$